

# MEAN CURVATURE 1 SURFACES IN HYPERBOLIC 3-SPACE WITH LOW TOTAL CURVATURE I

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*Dedicated to Katsuhiro Shiohama on the occasion of his sixtieth birthday.*

**ABSTRACT.** A complete surface of constant mean curvature 1 (CMC-1) in hyperbolic 3-space with constant curvature  $-1$  has two natural notions of “total curvature”—one is the *total absolute curvature* which is the integral over the surface of the absolute value of the Gaussian curvature, and the other is the *dual total absolute curvature* which is the total absolute curvature of the dual CMC-1 surface. In this paper, we completely classify CMC-1 surfaces with dual total absolute curvature at most  $4\pi$ . Moreover, we give new examples and partially classify CMC-1 surfaces with dual total absolute curvature at most  $8\pi$ .

With the developments of the last decade on constant mean curvature 1 (CMC-1) surfaces in hyperbolic 3-space  $H^3$  (the complete simply-connected 3-manifold of constant sectional curvature  $-1$ ), and with so many examples now known, it is a natural next step to classify all such surfaces with low total absolute curvature.

As CMC-1 surfaces in  $H^3$  share quite similar properties with minimal surfaces in Euclidean 3-space  $\mathbf{R}^3$ , let us first comment that the total absolute curvature of a minimal surface in  $\mathbf{R}^3$  is equal to the area (counted with multiplicity) of the Gauss image of the surface, and that complete minimal surfaces in  $\mathbf{R}^3$  with total curvature at most  $8\pi$  have been classified. (See Lopez [6] and also Table 2.) Furthermore, as the Gauss map of a complete conformally parametrized minimal surface is holomorphic, and has a well-defined limit at each end when the surface has finite total curvature, the area of the Gauss image must be an integer multiple of  $4\pi$ .

The question of classifying low total curvature CMC-1 surfaces in  $H^3$  is analogous—however, unlike minimal surfaces in  $\mathbf{R}^3$ , CMC-1 surfaces in  $H^3$  have two Gauss maps: the hyperbolic Gauss map  $G$  and the secondary Gauss map  $g$ . So there are two ways to pose the question in  $H^3$ , with two very different answers. One way is to consider the true total absolute curvature, which is the area of the image of  $g$ , but since  $g$  might not be single-valued on the surface, the total curvature might not be an integer multiple of  $4\pi$ . This allows for many more possibilities and makes the problem more difficult than for minimal surfaces in  $\mathbf{R}^3$ . The authors take up this question in a separate paper [13].

The second way, which is the theme of this paper, is to study the area of the image of  $G$ , which we call the *dual* total absolute curvature, as it is the true total curvature of the dual CMC-1 surface (which we define in Section 1) in  $H^3$ . This way has the advantage that  $G$  is single-valued on the surface, and so the dual total

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curvature is always an integer multiple of  $4\pi$ , like the case of minimal surfaces in  $\mathbf{R}^3$ . Furthermore, the dual total curvature satisfies not only the Cohn-Vossen inequality, but also the hyperbolic analogue of the Osserman inequality (which cannot be said about the true total curvature) [19, 23] (see also (2.1) in Section 2). So the dual total curvature shares more properties with the total curvature of minimal surfaces in  $\mathbf{R}^3$ , motivating our interest in it.

In this paper, we classify CMC-1 surfaces with dual total absolute curvature at most  $4\pi$ , and we go much of the way toward classifying CMC-1 surfaces with dual total absolute curvature at most  $8\pi$  (as a first step to a full classification of the  $8\pi$  case). In Section 1, we give a summary of the results, and in Section 2 we give preliminaries for the latter sections. The classification of CMC-1 surfaces with dual total absolute curvature less than or equal to  $4\pi$  is given in Section 3. Surfaces with dual total absolute curvature  $8\pi$  are discussed in Section 4 — and there we find new examples, we classify certain cases, and we show nonexistence in certain other cases. In Section 5, from deformations of corresponding minimal surfaces in  $\mathbf{R}^3$ , we produce two classes of new CMC-1 surfaces with dual total absolute curvature  $8\pi$ . For the readers' convenience, we attach Appendix A to explain the computation of log-term coefficients of second order linear ordinary differential equations with regular singularities.

## 1. SUMMARY OF THE RESULTS

To state our results precisely, we begin with some notations. Let  $f: M \rightarrow H^3$  be a complete conformal CMC-1 immersion of a Riemann surface  $M$  into  $H^3$ . By Bryant's representation formula, there is a holomorphic null immersion  $F: \widetilde{M} \rightarrow \mathrm{SL}(2, \mathbf{C})$  such that  $f = FF^*$ , where  $\widetilde{M}$  is the universal cover of  $M$  and  $F^* = \overline{F}$ . ("null" means  $\det(F^{-1}dF) = 0$ .) Here, we consider  $H^3 = \mathrm{SL}(2, \mathbf{C})/\mathrm{SU}(2) = \{aa^* \mid a \in \mathrm{SL}(2, \mathbf{C})\}$  [1, 15]. We call  $F$  the *lift* of  $f$ , and  $F$  satisfies

$$(1.1) \quad dF = F \begin{pmatrix} g & -g^2 \\ 1 & -g \end{pmatrix} \frac{Q}{dg}$$

on  $\widetilde{M}$ , where  $g$  (the *secondary Gauss map*) is a meromorphic function defined on  $\widetilde{M}$  and  $Q$  (the *Hopf differential*) is a holomorphic 2-differential on  $M$ . Then the induced metric  $ds^2$  and complexification of the second fundamental form  $h$  are

$$ds^2 = (1 + |g|^2)^2 \left| \frac{Q}{dg} \right|^2, \quad h = -Q - \overline{Q} + ds^2.$$

By (1.1), the secondary Gauss map satisfies

$$g = -\frac{dF_{12}}{dF_{11}} = -\frac{dF_{22}}{dF_{21}}, \quad \text{where} \quad F(z) = \begin{pmatrix} F_{11}(z) & F_{12}(z) \\ F_{21}(z) & F_{22}(z) \end{pmatrix}.$$

The map  $g$  is determined uniquely up to a Möbius transformation  $g \mapsto a \star g$  by  $a \in \mathrm{SU}(2)$ , where, for general  $a = (a_{ij}) \in \mathrm{SL}(2, \mathbf{C})$ , we denote

$$a \star g := \frac{a_{11}g + a_{12}}{a_{21}g + a_{22}}.$$

The *hyperbolic Gauss map*  $G$  of  $f$  is defined by

$$G = \frac{dF_{11}}{dF_{21}} = \frac{dF_{12}}{dF_{22}},$$

which can be interpreted as stereographic projection of the endpoints in the sphere at infinity of  $H^3$  of the oriented normal geodesics emanating from the surface. In particular,  $G$  is a meromorphic function on  $M$ .

The inverse matrix  $F^{-1}$  is also a holomorphic null immersion, and produces a new CMC-1 immersion  $f^\# = F^{-1}(F^{-1})^*: \widetilde{M} \rightarrow H^3$ , called the *dual* of  $f$  [19]. The induced metric  $ds^{2\#}$  and the Hopf differential  $Q^\#$  of  $f^\#$  are

$$(1.2) \quad ds^{2\#} = (1 + |G|^2)^2 \left| \frac{Q}{dG} \right|^2, \quad Q^\# = -Q.$$

So  $ds^{2\#}$  and  $Q^\#$  are well-defined on  $M$  itself, even though  $f^\#$  might be defined only on  $\widetilde{M}$ . This duality between  $f$  and  $f^\#$  interchanges the roles of the hyperbolic Gauss map  $G$  and secondary Gauss map  $g$ . In particular, one has

$$(1.3) \quad dF \cdot F^{-1} = -(F^{-1})^{-1} d(F^{-1}) = \begin{pmatrix} G & -G^2 \\ 1 & -G \end{pmatrix} \frac{Q}{dG}.$$

Hence  $dFF^{-1}$  is single-valued on  $M$ , whereas  $F^{-1}dF$  generally is not.

Since  $ds^{2\#}$  is single-valued on  $M$ , we can define the *dual total absolute curvature*

$$\text{TA}(f^\#) := \int_M (-K^\#) dA^\#,$$

where  $K^\#$  ( $\leq 0$ ) and  $dA^\#$  are the Gaussian curvature and area element of  $ds^{2\#}$ , respectively. As

$$(1.4) \quad d\sigma^{2\#} := (-K^\#) ds^{2\#} = \frac{4 dG d\overline{G}}{(1 + |G|^2)^2}$$

is a pseudo-metric of constant curvature 1 with developing map  $G$ ,  $\text{TA}(f^\#)$  is the area of the image of  $G$  on  $\mathbf{CP}^1 = S^2$ . The following assertion is important for us:

**Lemma 1.1** ([19, 22]). *The Riemannian metric  $ds^{2\#}$  is complete (resp. nondegenerate) if and only if  $ds^2$  is complete (resp. nondegenerate).*

So from now on, we suppose  $f$  is complete and has  $\text{TA}(f^\#) < +\infty$ . By Lemma 1.1, the conformal metric  $ds^{2\#}$  is complete. As  $\text{TA}(f^\#) < +\infty$ ,  $M$  is biholomorphic to a compact Riemann surface  $\overline{M}_\gamma$  of some genus  $\gamma$  with finitely many points excluded [8, Theorem 9.1]:

$$(1.5) \quad M = \overline{M}_\gamma \setminus \{p_1, \dots, p_n\} \quad (p_1, \dots, p_n \in \overline{M}_\gamma).$$

The points  $p_j$  are called the *ends* of the immersion  $f$ .

If  $G$  has an essential singularity at any end  $p_j$ , then  $\text{TA}(f^\#) = +\infty$ , since  $\text{TA}(f^\#)$  is the area of  $G(M)$  in  $\mathbf{CP}^1 = S^2$ . Since we have assumed  $\text{TA}(f^\#) < +\infty$ ,  $G$  is meromorphic on all of  $\overline{M}_\gamma$ . In particular,  $\text{TA}(f^\#) = 4\pi \deg G \in 4\pi\mathbf{Z}$ .

Since the dual immersion has finite total curvature, the Hopf differential  $Q^\# = -Q$  can be extended to  $\overline{M}_\gamma$  as a meromorphic 2-differential [1, Proposition 5]. Let

$$d_j = \text{ord}_{p_j} Q = \text{order of } Q \text{ at the end } p_j$$

for each  $j = 1, \dots, n$ . We say that  $f$  is a surface of *type*  $\mathbf{\Gamma}(d_1, \dots, d_n)$  if  $M = \overline{M}_\gamma \setminus \{p_1, \dots, p_n\}$  and  $Q$  has order  $d_j$  at each end  $p_j$ . We use  $\mathbf{\Gamma}$  because it is the capitalized letter corresponding to  $\gamma$ , the genus of  $\overline{M}_\gamma$ . For instance, the class  $\mathbf{I}(-4)$  (resp.  $\mathbf{O}(-2, -3)$ ) means the class of surfaces of genus 1 (resp. genus 0) with 1 end (resp. 2 ends) so that  $Q$  has a pole of order 4 at the single end (resp. a pole of order

Type	TA( $f^\#$ )	Reducibility	Status	c.f.
$\mathbf{O}(0)$	0	$\mathcal{H}^3$ -reducible	classified <sup>0</sup>	Horosphere
$\mathbf{O}(-4)$	$4\pi$	$\mathcal{H}^3$ -reducible	classified	Duals of Enneper cousins [10, Example 5.4]
$\mathbf{O}(-2, -2)$	$4\pi$	reducible	classified	Catenoid cousins and warped catenoid cousins with embed- ded ends [1, Example 2],[15],[13]
$\mathbf{O}(-5)$	$8\pi$	$\mathcal{H}^3$ -reducible	classified	Theorem 4.14
$\mathbf{O}(-6)$	$8\pi$	$\mathcal{H}^3$ -reducible	classified	Theorem 4.14
$\mathbf{O}(-2, -2)$	$8\pi$	reducible	classified	Double covers of catenoid cousins and warped catenoid cousins with $m = 2$ in [15, Theorem 6.2],[13]
$\mathbf{O}(-1, -4)$	$8\pi$	$\mathcal{H}^3$ -reducible	classified <sup>0</sup>	Theorem 4.13
$\mathbf{O}(-2, -3)$	$8\pi$	$\mathcal{H}^1$ -reducible	classified	Theorems 4.11, 4.12
$\mathbf{O}(-2, -4)$	$8\pi$	$\mathcal{H}^1$ -reducible	classified	Theorem 4.9
		$\mathcal{H}^3$ -reducible	classified	Theorem 4.10
$\mathbf{O}(-3, -3)$	$8\pi$	reducible	existence	Proposition 4.8
$\mathbf{O}(-1, -1, -2)$	$8\pi$	$\mathcal{H}^3$ -reducible	classified <sup>0</sup>	Theorem 4.7
$\mathbf{O}(-1, -2, -2)$	$8\pi$	$\mathcal{H}^1$ -reducible	classified	Theorem 4.5
		$\mathcal{H}^3$ -reducible	classified	Theorem 4.6
$\mathbf{O}(-2, -2, -2)$	$8\pi$	irreducible	classified	[20, Theorem 2.6]
		$\mathcal{H}^1$ -reducible	existence <sup>+</sup>	Example 4.3
		$\mathcal{H}^3$ -reducible	existence <sup>+</sup>	Example 4.4
$\mathbf{I}(-3)$	$8\pi$		unknown	
$\mathbf{I}(-4)$	$8\pi$		existence	Proposition 4.2
$\mathbf{I}(-1, -1)$	$8\pi$		unknown <sup>+</sup>	Proposition 4.1
$\mathbf{I}(-2, -2)$	$8\pi$		existence	Genus 1 catenoid cousins [9]

TABLE 1. CMC-1 surfaces in  $H^3$  with  $\text{TA}(f^\#) \leq 8\pi$ . (The corresponding results for minimal surfaces in  $\mathbf{R}^3$  are shown in Table 2.)

2 at one end and order 3 at the other). Then our results are shown in Table 1. In the table,

- *classified* means the complete list of the surfaces in such a class is known (and this means not only that we know all the possibilities for the form of the data  $(G, Q)$ , but that we also know exactly for which  $(G, Q)$  the period problems of the immersions are solved).
- *classified*<sup>0</sup> means there exists a unique surface (up to isometries of  $H^3$  and deformations that come from its reducibility).
- *existence* means that examples exist, but they are not yet classified.
- *existence*<sup>+</sup> means that all possibilities for the data  $(G, Q)$  are determined in this paper, but the period problems are solved only for special cases.
- *unknown* means that neither existence nor non-existence is known yet.
- *unknown*<sup>+</sup> means that all possibilities for the data  $(G, Q)$  are determined in this paper, but the period problems are still unsolved.

Any class and type of reducibility not listed in Table 1 cannot contain surfaces with  $\text{TA}(f^\#) \leq 8\pi$ . For example, any irreducible or  $\mathcal{H}^3$ -reducible surface of type

$\mathbf{O}(-2, -3)$  must have dual total absolute curvature at least  $12\pi$ . (See Section 2 for the definitions of irreducibility,  $\mathcal{H}^1$ -reducibility, and  $\mathcal{H}^3$ -reducibility.)

Type	TA	The surface	c.f.
$\mathbf{O}(0)$	0	Plane	
$\mathbf{O}(-4)$	$4\pi$	Enneper's surface	
$\mathbf{O}(-5)$	$8\pi$		[6, Theorem 6]
$\mathbf{O}(-6)$	$8\pi$		[6, Theorem 6]
$\mathbf{O}(-2, -2)$	$4\pi$	Catenoid	
	$8\pi$	Double cover of the catenoid	
$\mathbf{O}(-1, -3)$	$8\pi$		[6, Theorem 5]
$\mathbf{O}(-2, -3)$	$8\pi$		[6, Theorem 4, 5]
$\mathbf{O}(-2, -4)$	$8\pi$		[6, Theorem 5]
$\mathbf{O}(-3, -3)$	$8\pi$		[6, Theorem 4]
$\mathbf{O}(-1, -2, -2)$	$8\pi$		[6, Theorem 5]
$\mathbf{O}(-2, -2, -2)$	$8\pi$		[6, Theorem 5]
$\mathbf{I}(-4)$	$8\pi$	Chen-Gackstatter surface	[6, Theorem 5], [2]

TABLE 2. The classification of complete minimal surfaces in  $\mathbf{R}^3$  with  $\text{TA} \leq 8\pi$  ([6]), for comparison with Table 1.

## 2. PRELIMINARIES

Before we begin proving the results, we prepare some fundamental properties and tools, which will play important roles in the latter sections.

**Analogue of the Osserman inequality.** The second and third authors showed [19]:

$$(2.1) \quad \frac{1}{2\pi} \text{TA}(f^\#) \geq -\chi(M) + n = 2(\gamma + n - 1) .$$

Moreover, equality holds exactly when all the ends are embedded: This follows by noting that equality is equivalent to all ends being regular and embedded ([19]), and that any embedded end must be regular (proved recently by Collin, Hauswirth and Rosenberg [4]).

**Formulas for  $\text{TA}(f^\#)$ .** Let  $\mu_j^\# \in \mathbf{Z}$  be the branching order of  $G$  at the end  $p_j$  for each  $j = 1, \dots, n$ . Since  $G$  is a  $(\mu_j^\# + 1)$ -to-1 mapping in a neighborhood of  $p_j$ ,

$$(2.2) \quad \mu_j^\# \leq \deg G - 1 = \frac{1}{4\pi} \text{TA}(f^\#) - 1 .$$

The umbilic points of  $f$  are the zeroes of  $Q = -Q^\#$ , which are also the umbilic points of  $f^\#$ . Moreover, the order of  $Q$  equals the branching order of  $G$  at each point in  $M$ , since  $ds^{2\#}$  in (1.2) is non-degenerate. Let  $q_1, \dots, q_k$  be the umbilic points of  $f$  and set

$$(2.3) \quad \xi_l := \text{ord}_{q_l} Q = [\text{the branching order of } G \text{ at } q_l] \quad (l = 1, \dots, k) .$$

The pseudometric  $d\sigma^{2\#}$  in (1.4) is said to have order  $\beta$  at  $p$  if it is asymptotic to  $|z - z(p)|^{2\beta} dz d\bar{z}$ , where  $z$  is a complex coordinate around  $p$ . Then the branching

order of  $G$  is equal to the order of the metric  $d\sigma^{2\#}$  in (1.4), the Gauss-Bonnet theorem implies that

$$(2.4) \quad \frac{1}{2\pi} \text{TA}(f^\#) = \chi(\overline{M}_\gamma) + \sum_{j=1}^n \mu_j^\# + \sum_{l=1}^k \xi_l ,$$

where  $\chi(\cdot)$  is the Euler characteristic. (This also follows from the Riemann-Hurwitz formula, since  $\overline{M}_\gamma$  is a branched cover of  $S^2$  via the map  $G$ .)

Since  $Q$  is a meromorphic 2-differential, the total order of  $Q$  satisfies

$$(2.5) \quad \sum_{l=1}^k \xi_l + \sum_{j=1}^n d_j = -2\chi(\overline{M}_\gamma) .$$

By (2.4) and (2.5), we have

$$(2.6) \quad \frac{1}{2\pi} \text{TA}(f^\#) = -\chi(\overline{M}_\gamma) + \sum_{j=1}^n (\mu_j^\# - d_j) = 2\gamma - 2 + \sum_{j=1}^n (\mu_j^\# - d_j) .$$

Completeness of the metric  $ds^{2\#}$  at  $p_j$  implies  $\mu_j^\# - d_j \geq 1$ . However, the case  $\mu_j^\# - d_j = 1$  cannot occur ([19, Lemma 3]), so

$$(2.7) \quad \mu_j^\# - d_j \geq 2 .$$

**Effects of transforming the lift  $F$ .** Here we consider the change  $\hat{F} = aFb^{-1}$  of the lift  $F$ , where  $a, b \in \text{SL}(2, \mathbf{C})$ . Then  $\hat{F}$  is also a holomorphic null immersion, and the hyperbolic Gauss map  $\hat{G}$ , the secondary Gauss map  $\hat{g}$  and the Hopf differential  $\hat{Q}$  of  $\hat{F}$  are given by (see [17])

$$(2.8) \quad \hat{G} = a \star G, \quad \hat{g} = b \star g, \quad \hat{Q} = Q .$$

In particular, the change  $\hat{F} = aF$  moves the surface by a rigid motion of  $H^3$ , and does not change  $g$  and  $Q$ . By choosing a suitable rigid motion  $a \in \text{SL}(2, \mathbf{C})$  of the surface in  $H^3$ , we shall frequently use the following change of the hyperbolic Gauss map to simplify its expression:

$$(2.9) \quad \hat{G} = a \star G = \frac{a_{11}G + a_{12}}{a_{21}G + a_{22}}, \quad (a_{ij})_{i,j=1,2} \in \text{SL}(2, \mathbf{C}) .$$

**The Schwarzian derivative relation.** A direct computation implies that the secondary Gauss map  $g$  depends on  $G$  and  $Q$  as follows ([15]):

$$(2.10) \quad S(g) - S(G) = 2Q ,$$

where

$$S(g) = \left[ \left( \frac{g''}{g'} \right)' - \frac{1}{2} \left( \frac{g''}{g'} \right)^2 \right] dz^2 \quad \left( ' = \frac{d}{dz} \right)$$

is the *Schwarzian derivative* of  $g$ . Here,  $z$  is a complex coordinate of  $\overline{M}_\gamma$ .

**SU(2)-monodromy conditions.** Here we recall from [10] the construction of CMC-1 surfaces with given hyperbolic Gauss map  $G$  and Hopf differential  $Q$ , which will play a crucial role in this paper. Let  $\overline{M}_\gamma$  be a compact Riemann surface and  $M := \overline{M}_\gamma \setminus \{p_1, \dots, p_n\}$ . Let  $G$  and  $Q$  be a meromorphic function and meromorphic 2-differential on  $\overline{M}_\gamma$ . The pair  $(G, Q)$  must satisfy the following two compatibility conditions:

$$(2.11) \quad \text{For all } q \in M, \text{ord}_q Q \text{ is equal to the branching order of } G, \text{ and}$$

$$(2.12) \quad \text{for each end } p_j, \mu_j^\# - d_j \geq 2.$$

The first condition implies that the metric  $ds^{2\#}$  is (and hence  $ds^2$  is also, by Lemma 1.1) non-degenerate at  $q \in M$ . The second condition implies that the metric  $ds^{2\#}$  is complete (and hence  $ds^2$  is also, again by Lemma 1.1) at  $p_j \in \overline{M}_\gamma$  ( $j = 1, \dots, n$ ).

For such a pair  $(G, Q)$ , a solution  $g$  of equation (2.10) has singularities at the branch points of  $G$  (umbilic points or ends) and the poles of  $Q$  (ends). However, regardless of whether  $q \in M$  is a regular or umbilic point,  $ds^{2\#}$  and  $Q^\#$  as in (1.2) give a (non-degenerate) Riemann metric and holomorphic 2-differential in a neighborhood  $U_q \subset M$  of  $q$ . Then, by the fundamental theorem of surfaces, there exists a CMC-1 immersion  $f^\#$  of  $U_q$  into  $H^3$  with induced metric  $ds^{2\#}$  and Hopf differential  $Q^\#$ . So the hyperbolic Gauss map  $g$  of  $f^\#$ , which is a solution of (2.10), is a well-defined meromorphic function on  $U_q$ . Since the solution of (2.10) is unique up to Möbius transformations  $g \mapsto a \star g$  ( $a \in \text{SL}(2, \mathbb{C})$ ), for any solution  $g$  of (2.10) defined on the universal cover  $\tilde{M}$  of  $M$ , there exists a representation

$$\rho_g: \pi_1(M) \rightarrow \text{PSL}(2) \quad \text{such that} \quad g \circ \tau^{-1} = \rho_g(\tau) \star g$$

for each covering transformation  $\tau \in \pi_1(M)$ .

We now consider when the dual  $f = (f^\#)^\#$  (with data  $(G, Q)$ ) of  $f^\#$  is well-defined on  $M$ . Choosing  $F$  so that  $F^{-1}$  is a lift of  $f^\#$  (and then also  $(F^{-1})^{-1} = F$  is a lift of  $(f^\#)^\# = f$ ), and noting that the representation  $\rho_g: \pi_1(M) \rightarrow \text{PSL}(2, \mathbb{C})$  can be lifted into  $\text{SL}(2, \mathbb{C})$  [10], (2.8) implies

$$(2.13) \quad F^{-1} \circ \tau^{-1} = \rho_g(\tau) F^{-1}$$

for each  $\tau \in \pi_1(M)$ . Thus

$$(2.14) \quad f \circ \tau^{-1} = (F \circ \tau^{-1})(F \circ \tau^{-1})^* = F(\rho_g(\tau))^{-1}((\rho_g(\tau))^{-1})^* F^*,$$

and so  $f$  is well-defined on  $M$  if  $\rho_g(\tau) \in \text{SU}(2)$  for all  $\tau \in \pi_1(M)$ . This is the crux of the following Lemma 2.1. Before stating this lemma, we need a definition:

**Definition 1.** A CMC-1 immersion  $f: M \rightarrow H^3$  is *reducible* if  $\{\rho_g(\tau)\}_{\tau \in \pi_1(M)}$  are simultaneously diagonalizable (i.e. if there exists a  $P \in \text{PSL}(2, \mathbb{C})$  such that  $P\rho_g(\tau)P^{-1}$  is diagonal for all  $\tau \in \pi_1(M)$ ). If  $f$  is not reducible, it is called *irreducible*. When  $f$  is reducible, it is either  $\mathcal{H}^3$ -reducible or  $\mathcal{H}^1$ -reducible [10], and  $f$  is called  $\mathcal{H}^3$ -reducible if  $\{\rho_g(\tau)\}_{\tau \in \pi_1(M)}$  are all the identity, and is called  $\mathcal{H}^1$ -reducible otherwise.

Clearly  $f$  is  $\mathcal{H}^3$ -reducible if and only if the lift  $F$  itself is single-valued on  $M$ , by (2.13). The name  $\mathcal{H}^1$ -reducibility (resp.  $\mathcal{H}^3$ -reducibility) comes from the fact that the surface has exactly a 1 (resp. 3) dimensional deformation through surfaces preserving  $G$  and  $Q$  and the mean curvature, which is identified with the 1 (resp. 3) dimensional hyperbolic space  $\mathcal{H}^1$  (resp.  $\mathcal{H}^3$ ) [10]. On the other hand, if

$f$  is irreducible,  $f$  has no deformation preserving mean curvature and  $(G, Q)$  (see [17, 10]).

**Lemma 2.1** ([17]). *Let  $G$  and  $Q$  be a meromorphic function and a meromorphic 2-differential on  $\overline{M}_\gamma$  satisfying (2.11) and (2.12). Assume  $g$  is a solution of (2.10) such that the image of  $\rho_g$  lies in  $\text{PSU}(2)$ . Then there exists a complete CMC-1 immersion  $f : M \rightarrow H^3$  with hyperbolic Gauss map  $G$ , Hopf differential  $Q$ , and secondary Gauss map  $g$ .*

*If  $f$  is irreducible, then  $f$  is the unique surface with data  $(G, Q)$ . If  $f$  is  $\mathcal{H}^1$ -reducible (resp.  $\mathcal{H}^3$ -reducible), then there exists exactly a 1 (resp. 3) parameter family of CMC-1 surfaces with data  $(G, Q)$ .*

In the case that  $M$  is of genus  $\gamma = 0$  with at most two ends,  $f$  is reducible, as the fundamental group is commutative. More generally, for the case  $\gamma = 0$  with  $n$  ends, by Lemma 2.1 and the theory of linear ordinary differential equations (see Appendix A), we have:

**Proposition 2.2.** *Let  $\overline{M}_0 = \mathcal{C} \cup \{\infty\}$  and  $M = \overline{M}_0 \setminus \{p_1, \dots, p_n\}$  with  $p_1, \dots, p_{n-1} \in \mathcal{C}$ . Let  $G$  and  $Q$  be a meromorphic function and a meromorphic 2-differential on  $\mathcal{C} \cup \{\infty\}$  satisfying (2.11) and (2.12). Consider the linear ordinary differential equation*

$$(E.0) \quad \frac{d^2 u}{dz^2} + r(z)u = 0,$$

where  $r(z) dz^2 := (S(G)/2) + Q$ . Suppose  $n \geq 2$ , and also  $d_j = \text{ord}_{p_j} Q \geq -2$  and the indicial equation of (E.0) at  $z = p_j$  has the two roots  $\lambda_1^{(j)}, \lambda_2^{(j)}$  and log-term coefficient  $c_j$ , for  $j = 1, 2, \dots, n-1$ .

- (1) *Suppose that  $\lambda_1^{(j)} - \lambda_2^{(j)} \in \mathbf{Z}^+$  and  $c_j = 0$  for  $j \leq n-1$ . Then there is exactly a 3-parameter family of complete conformal CMC-1 immersions of  $M$  into  $H^3$  with hyperbolic Gauss map  $G$  and Hopf differential  $Q$ . Moreover, such surfaces are  $\mathcal{H}^3$ -reducible.*
- (2) *Suppose that  $\lambda_1^{(j)} - \lambda_2^{(j)} \in \mathbf{Z}^+$  and  $c_j = 0$  for  $j \leq n-2$ , and that  $\lambda_1^{(n-1)} - \lambda_2^{(n-1)} \in \mathbf{R} \setminus \mathbf{Z}$ . Then there exists exactly a 1-parameter family of complete conformal CMC-1 immersions of  $M$  into  $H^3$  with hyperbolic Gauss map  $G$  and Hopf differential  $Q$ . Moreover, such surfaces are  $\mathcal{H}^1$ -reducible.*

Here we denoted by  $\mathbf{Z}^+$  the set of positive integers.

The ordinary differential equation (E.0) has also been applied in [7] for constructing certain classes of  $\mathcal{H}^3$ -reducible CMC-1 surfaces.

*Proof.* The general theory of Schwarzian derivatives shows ([21, Chapter 4]) that for a linearly independent pair  $u_1, u_2$  of solutions of (E.0), the function  $g := u_1/u_2$  satisfies (2.10). Conversely, any function  $g$  satisfying  $S(g) = r(z) dz^2$  is obtained in this way.

If  $\lambda_1^{(j)} - \lambda_2^{(j)} = m \in \mathbf{Z}^+$  and  $c_j = 0$ , then there is a fundamental system of solutions of (E.0) in a neighborhood of  $p_j$  of the form

$$(2.15) \quad u_1 = (z - p_j)^{\lambda_1^{(j)}} \varphi_1(z), \quad u_2 = (z - p_j)^{\lambda_1^{(j)} - m} \varphi_2(z),$$

where  $\varphi_1(z)$  and  $\varphi_2(z)$  are holomorphic and nonzero at  $z = p_j$ . Then  $g := u_1/u_2$  satisfies

$$(2.16) \quad g \circ \tau_j^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \star g ,$$

where  $\tau_j$  is the covering transformation which corresponds to a small loop around  $z = p_j$ , implying  $\rho_g(\tau_j) = \text{identity}$ . So for case (1), we have  $\rho_g(\tau_j) = \text{identity}$  for all  $j = 1, \dots, n-1$ , and therefore also for  $j = n$ , which implies that  $g$  is a meromorphic function on  $\mathcal{C} \cup \{\infty\}$ . By Lemma 2.1, there exists a conformal CMC-1 immersion  $f_a$  on  $M$  with the secondary Gauss map  $a \star g$  for all  $a \in \text{SL}(2, \mathcal{C})$ . If  $a \in \text{SU}(2)$ , then  $f_a$  coincides with  $f_{\text{identity}}$  by (2.14), so we have that the 3-parameter family  $(f_{[a]})_{[a] \in \text{SL}(2, \mathcal{C})/\text{SU}(2)}$  are complete conformal CMC-1 immersions with hyperbolic Gauss map  $G$  and Hopf differential  $Q$ .

We remark here that if  $\lambda_1^{(j)} - \lambda_2^{(j)} = m \in \mathbf{Z}^+$  and  $c_j \neq 0$ , then the monodromy matrix  $\rho_g(\tau_j)$  defined by  $g \circ \tau_j^{-1} = \rho_g(\tau_j) \star g$  is not diagonalizable and is not even in  $\text{SU}(2)$ . So any CMC-1 immersion on  $\widetilde{M}$  (with  $G$  and  $Q$ ) cannot be well-defined on  $M$  when some  $c_j \neq 0$ .

Next we consider case (2), that is  $\lambda_1^{(n-1)} - \lambda_2^{(n-1)} \notin \mathbf{Z}$ . There exists a fundamental system of solutions of (E.0) of the form

$$(2.17) \quad u_1 = (z - p_{n-1})^{\lambda_1^{(n-1)}} \varphi_1(z) , \quad u_2 = (z - p_{n-1})^{\lambda_2^{(n-1)}} \varphi_2(z) ,$$

where  $\varphi_1(z)$  and  $\varphi_2(z)$  are holomorphic and nonzero at  $z = p_{n-1}$ . When  $\tau_{n-1}$  is the covering transformation induced from a small loop about  $z = p_{n-1}$ ,  $g := u_1/u_2$  satisfies

$$(2.18) \quad g \circ \tau_{n-1}^{-1} = \begin{bmatrix} e^{\pi i(\lambda_1^{(n-1)} - \lambda_2^{(n-1)})} & 0 \\ 0 & e^{\pi i(\lambda_2^{(n-1)} - \lambda_1^{(n-1)})} \end{bmatrix} \star g .$$

In particular,  $\rho_g(\tau_{n-1}) \in \text{SU}(2)$ . On the other hand, in the proof of (1), we have seen that  $\rho_g(\tau_j) = \text{identity}$  for  $j \in (1, \dots, n-2)$ . Hence  $\rho_g(\tau_j) \in \text{SU}(2)$  and are diagonal matrices for all  $j \in (1, \dots, n)$ , and we are in the  $\mathcal{H}^1$ -reducible case. Note that this remains true when  $g$  is replaced by

$$sg(z) = a(s) \star g , \quad \text{where} \quad a(s) := \begin{bmatrix} \sqrt{s} & 0 \\ 0 & 1/\sqrt{s} \end{bmatrix} , \quad \text{with} \quad s \in \mathbf{R}^+ ,$$

where  $\mathbf{R}^+$  is the set of positive reals. So we have a one-parameter family of complete conformal CMC-1 immersions with hyperbolic Gauss map  $G$  and Hopf differential  $Q$  and secondary Gauss maps  $sg$  for  $s \in \mathbf{R}^+$ . ( $s_1 g$  and  $s_2 g$  for  $s_1 \neq s_2$  will not produce equivalent surfaces, as  $a(s_1)(a(s_2))^{-1} \notin \text{SU}(2)$ .) Furthermore, Lemma 2.1 implies there is *only* a one-parameter family of CMC-1 immersions with data  $(G, Q)$ .  $\square$

By (1.3), we have

$$(F^{-1})^{-1} d(F^{-1}) = \begin{pmatrix} g^\# & -g^{\#2} \\ 1 & -g^\# \end{pmatrix} \omega^\# ,$$

where

$$g^\# = G , \quad \omega^\# = -\frac{Q}{dG} .$$

By Lemma 2.1 of [15] (replacing  $F$  with  $F^{-1}$ ), we have that  $X = F_{21}(z), F_{22}(z)$  satisfies the equation

$$(E.1)^\# \quad X'' - (\log(\hat{\omega}^\#))' X' + \hat{Q}X = 0 ,$$

and  $Y = F_{11}(z), F_{12}(z)$  satisfies the equation

$$(E.2)^\# \quad Y'' - (\log(G^2 \hat{\omega}^\#))' Y' + \hat{Q}Y = 0 ,$$

where  $Q(z) = \hat{Q}(z)dz^2$  and  $\omega^\# = \hat{\omega}^\#(z)dz$ . (We call them  $(E.1)^\#$  and  $(E.2)^\#$  because they are the dual versions of equations (E.1) and (E.2) in [15].) These two equations have been shown in [23] as a modification of the corresponding equations in [15]. As we will see later, equations  $(E.1)^\#$  and  $(E.2)^\#$  are sometimes more convenient than equation (E.0) for solving monodromy problems. In fact, we will have use for the following lemma:

**Lemma 2.3.** *Let  $G$  and  $Q$  be a meromorphic function and a holomorphic 2-differential on  $D^* = \{z \in \mathbf{C}; 0 < |z| < 1\}$  such that the metric  $ds^{2\#}$  defined by (1.2) is positive definite on  $D^*$  and complete at 0. Assume  $\text{ord}_{z=0} Q \geq -2$  and  $Q$  is not identically zero. Then the following three conditions are all equivalent.*

- (1) *The difference of the solutions of the indicial equation of  $(E.1)^\#$  at  $z = 0$  is a positive integer and the log-term coefficient of  $(E.1)^\#$  vanishes.*
- (2) *The difference of the solutions of the indicial equation of  $(E.2)^\#$  at  $z = 0$  is a positive integer and the log-term coefficient of  $(E.2)^\#$  vanishes.*
- (3) *The difference of the solutions of the indicial equation of (E.0) at  $z = 0$  is a positive integer and the log-term coefficient of (E.0) vanishes.*

*Proof.* The hyperbolic Gauss map of the dual surface  $f^\# = F^{-1}(F^{-1})^*$  is equal to the secondary Gauss map  $g$  of  $f = FF^*$ . Thus conditions (1) and (2) are equivalent to the condition that  $g$  is single valued at  $z = 0$ , by Lemma 2.2 of [15]. On the other hand, as seen in the proof of Proposition 2.2, condition (3) is also equivalent to the condition that  $g$  is single valued at  $z = 0$ .  $\square$

Here is a natural place to include the next lemma, which we shall use in the sequel, [13], to this paper.

**Lemma 2.4.** *With the same assumptions as in Lemma 2.3, the following three conditions are all equivalent.*

- (1) *The difference of the solutions of the indicial equation of  $(E.1)^\#$  at  $z = 0$  is a real number.*
- (2) *The difference of the solutions of the indicial equation of  $(E.2)^\#$  at  $z = 0$  is a real number.*
- (3) *The difference of the solutions of the indicial equation of (E.0) at  $z = 0$  is a real number.*

*Proof.* We write

$$G(z) = z^\mu \hat{G}(z) , \quad \omega^\#(z) = z^\nu \hat{\omega}^\#(z)dz ,$$

where  $\hat{G}$  and  $\hat{\omega}^\#$  are nonzero and holomorphic at  $z = 0$ , for some integers  $\mu$  and  $\nu$ .

If  $\text{ord}_{z=0} Q = -2$ , so  $\mu + \nu = -1$  and  $Q = (\theta z^{-2} + \dots)dz^2$  for some  $\theta \neq 0$ , then the difference of the solutions of the indicial equations is  $\sqrt{\mu^2 - 4\theta}$  in all three cases, hence the three statements are clearly equivalent.

If  $\text{ord}_{z=0} Q \geq -1$ , then the indicial equation in the first case (resp. second case, third case) is

$$t(t-1) - \nu t = 0, \quad \left( \text{resp. } t(t-1) - (2\mu + \nu)t = 0, \quad t(t-1) + \frac{1-\mu^2}{4} = 0 \right).$$

Hence the difference of the roots is  $|\nu + 1|$  (resp.  $|2\mu + \nu + 1|, |\mu|$ ), and so all three statements hold.  $\square$

### 3. THE CLASSIFICATION OF SURFACES WITH $\text{TA}(f^\#) \leq 4\pi$

We begin our consideration of classification with this simple case:

**Theorem 3.1.** *A complete CMC-1 immersion  $f$  with  $\text{TA}(f^\#) \leq 4\pi$  is congruent to one of the following:*

- (1) *a horosphere,*
- (2) *an Enneper cousin dual,  $(g, Q) = (\tan \sqrt{\theta} z, \theta dz^2)$  ( $\theta \in \mathbf{C} \setminus \{0\}$ ),*
- (3) *a catenoid cousin,*

$$(g, Q) = \left( az^\mu, \frac{1-\mu^2}{4z^2} dz^2 \right) \quad (a \in \mathbf{R}^+, \mu \in \mathbf{R}^+ \setminus \{1\}),$$

- (4) *a warped catenoid cousin that has a degree 1 hyperbolic Gauss map,*

$$(g, Q) = \left( az^l + b, \frac{1-l^2}{4z^2} dz^2 \right) \quad (a, b \in \mathbf{C} \setminus \{0\}, l \in \mathbf{Z}^+ \setminus \{1\}).$$

*Proof.* Since  $\text{TA}(f^\#) \in 4\pi\mathbf{Z}$ , we need to consider only the cases  $\text{TA}(f^\#) = 0$  and  $4\pi$ . If  $\text{TA}(f^\#) = 0$ , then the hyperbolic Gauss map is constant, so (1.4) implies  $K^\# \equiv 0$ . Thus  $f^\#$  is a totally umbilic CMC-1 immersion, so both  $f^\#$  and  $f$  are horospheres. So we consider the remaining case  $\text{TA}(f^\#) = 4\pi$ . Then  $G$  is meromorphic of degree 1 on  $\overline{M}_\gamma$ , which implies  $\gamma = 0$ . Hence we may choose  $\overline{M}_0 = \mathbf{C} \cup \{\infty\}$ , and by (2.9), we may assume  $G = z$ . Since  $G$  has no branch points, (2.3) implies there are no umbilic points, and (2.2) implies

$$(3.1) \quad \mu_j^\# = 0$$

at each end  $p_j$ . By (2.6) and (3.1) and the fact that  $\gamma = 0$ , we have

$$(3.2) \quad 2 = \frac{1}{2\pi} \text{TA}(f^\#) = -2 - \sum_{j=1}^n d_j.$$

By (2.7), we have  $2 \geq -2 + 2n$ , so  $n = 1$  or  $2$ .

**The case  $n = 1$ .** In this case, (3.2) implies  $d_1 = -4$ . We may put the end at  $p_1 = \infty$ , and then  $Q$  has a single pole of order 4 at  $\infty$  and no zeroes. Thus  $Q = \theta dz^2$  for some  $\theta \in \mathbf{C} \setminus \{0\}$ .

A CMC-1 surface in  $H^3$  with secondary Gauss map  $g = z$  and Hopf differential  $Q = \theta dz^2$  is called an Enneper cousin [1]. So a surface with data  $(G, Q) = (z, \theta dz^2)$  is the dual of an Enneper cousin [10, Example 5.4]. (Recall that dualizing switches the two Gauss maps, and changes the Hopf differential only by a sign.)

**The case  $n = 2$ .** In this case, (3.2) becomes  $4 = -d_1 - d_2$ . Then  $d_j = -2$  ( $j = 1, 2$ ), by (2.7). Hence the immersion  $f$  is a CMC-1 surface of genus 0 whose two ends must both be regular [15], and this type of surface is classified in [15]. In particular,  $f$  is in the case  $m = 1$  of Theorem 6.2 in [15]. So the surface is either a *catenoid cousin* [1, Example 2] or a *warped catenoid cousin* with embedded ends (the case  $m = 1$  in Theorem 6.2 in [15]).  $\square$

The warped catenoid cousins are described in detail in [13].

#### 4. SURFACES WITH $\text{TA}(f^\#) = 8\pi$

We now assume  $f$  has  $\text{TA}(f^\#) = 8\pi$ . Then, by (2.6) and (2.7),

$$(4.1) \quad 6 = 2\gamma + \sum_{j=1}^n (\mu_j^\# - d_j) \geq 2(\gamma + n)$$

holds. Thus the possible cases are

$$(\gamma, n) = (0, 1), \quad (0, 2), \quad (0, 3), \quad (1, 1), \quad (1, 2), \quad \text{and} \quad (2, 1).$$

Since  $\text{TA}(f^\#) = 8\pi$ ,  $G$  is meromorphic on  $\overline{M}_\gamma$  of degree 2. Hence (2.2) implies

$$(4.2) \quad \mu_j^\# \leq 1 \quad (j = 1, 2, \dots, n),$$

and at each umbilic point  $q_l$ ,

$$(4.3) \quad \xi_l = 1 \quad (l = 1, 2, \dots, k).$$

**The case  $(\gamma, n) = (2, 1)$ .** Since equality holds in (2.1), the single end  $p_1$  is embedded. By (4.1),  $\mu_1^\# - d_1 = 2$ . Thus the possible cases are

$$(\mu_1^\#, d_1) = (0, -2) \quad \text{or} \quad (1, -1),$$

by (4.2). If  $(\mu_1^\#, d_1) = (0, -2)$ , the end  $p_1$  is of type I in the sense of [11], so the flux about this end does not vanish [11, Proposition 2]. If  $(\mu_1^\#, d_1) = (1, -1)$ , then, since the end is embedded, Corollary 5 in [11] implies that the flux about the end again does not vanish. But non-vanishing flux at a single end contradicts the balancing formula [11, Theorem 1], so the case  $(\gamma, n) = (2, 1)$  does not occur.

**The case  $(\gamma, n) = (1, 2)$ .** In this case, (4.1) implies  $4 = (\mu_1^\# - d_1) + (\mu_2^\# - d_2)$ . By (2.7), we have  $\mu_j^\# - d_j = 2$  for  $j = 1, 2$ . Hence (4.2) implies

$$(\mu_j^\#, d_j) = (0, -2) \quad \text{or} \quad (1, -1) \quad (j = 1, 2).$$

Assume  $d_1 = -2$  and  $d_2 = -1$ . Then, by the transformation (2.9) if necessary, we may assume the hyperbolic Gauss map has a zero or pole at each end. In this case, the end  $p_1$  is regular of type I, and  $p_2$  is regular of type II in the sense of [11], contradicting Theorem 7 in [11]. Hence this case is impossible, leaving the two remaining possibilities:

$$(4.4) \quad (\mu_1^\#, d_1) = (\mu_2^\#, d_2) = (0, -2),$$

$$(4.5) \quad (\mu_1^\#, d_1) = (\mu_2^\#, d_2) = (1, -1).$$

For the case (4.4), the first author and Sato [9] constructed a one-parameter family of “genus one catenoid cousins”. Note that such surfaces cannot exist as minimal surfaces in  $\mathbf{R}^3$ , by Schoen’s result [14].



FIGURE 1. Two CMC-1 trinoids in  $H^3$ , which are surfaces of type  $\mathbf{O}(-2, -2, -2)$ , and a genus 1 catenoid cousin, which is a surface of type  $\mathbf{I}(-2, -2)$ , shown in the Poincaré model of  $H^3$ . Only one of two congruent pieces of the right-most two surfaces is shown, and the other half of each surface is the reflection (in the plane containing the boundary curves seen here) of the piece shown.

**Surfaces of type  $\mathbf{I}(-1, -1)$ .** For the case (4.5), we can determine the candidates of  $(G, Q)$  explicitly as follows (however, the period problem is unsolved and no example is known):

**Proposition 4.1.** *Let  $\overline{M}_1 = \mathbf{C}/\Gamma$ , where  $\Gamma$  is a lattice on  $\mathbf{C}$ , and assume there exists a CMC-1 immersion  $f: \overline{M}_1 \setminus \{p_1, p_2\} \rightarrow H^3$  with  $\text{TA}(f^\#) = 8\pi$  of type  $\mathbf{I}(-1, -1)$ . Then there exists a generating pair  $\{v_1, v_2\} \subset \mathbf{C}$  of  $\Gamma$  such that the hyperbolic Gauss map  $G$  and Hopf differential  $Q$  are given by*

$$(4.6) \quad G = \wp(z), \quad Q(z) = \theta \frac{\sigma(z - v_1/2)\sigma(z - v_2/2)}{\sigma(z)\sigma(z - (v_1 + v_2)/2)} dz^2 \quad (\theta \in \mathbf{C} \setminus \{0\}),$$

where  $\wp(z)$  is the Weierstrass  $\wp$ -function and  $\sigma$  is the entire function defined by

$$\sigma(z) := z \prod_{v \in \Gamma \setminus \{0\}} \left\{ \left(1 - \frac{z}{v}\right) e^{\frac{z}{v} + \frac{z^2}{2v^2}} \right\}.$$

*Proof.* In this case, the hyperbolic Gauss map  $G$  is of degree 2. Without loss of generality, we may assume that  $z = 0$  is an end of the surface. Moreover, by (2.9) we may assume that  $z = 0$  is a pole of  $G$ . As  $z = 0$  is a branch point of  $G$  (since  $\mu_j^\# = 1$ ),  $G$  has a pole of order 2 at  $z = 0$ . Up to a constant multiple, the function  $\wp(z)$  is uniquely characterized as a degree 2 meromorphic function on  $\mathbf{C}/\Gamma$  with a pole of order 2 at the origin [5]. Thus we have  $G(z) = c\wp(z)$ , and we can normalize  $c = 1$ , by (2.9).

Suppose  $\{v_1, v_2\}$  generates  $\Gamma$ . Then the branch points of  $\wp$  are  $0, v_1/2, v_2/2$  and  $(v_1 + v_2)/2$  modulo  $\Gamma$ , which are the ends and umbilic points. We assume  $0$  and  $(v_1 + v_2)/2$  are the ends. (If  $v_1/2$  is an end, for example, we may change the generator  $\Gamma$  to  $\{\tilde{v}_1 = v_1 - v_2, \tilde{v}_2 = v_2\}$ .) Thus the umbilic points are  $v_1/2$  and  $v_2/2$ .

Next we find the Hopf differential  $Q(z) = q(z) dz^2$ , using the following fact:

**Fact ([5]).** Let  $a_1, \dots, a_n$  and  $b_1, \dots, b_n$  be points in  $\mathbf{C}$  such that  $a_j \not\equiv b_k \pmod{\Gamma}$ ,  $j, k \in \{1, \dots, n\}$ , and  $\sum_{j=1}^n a_j = \sum_{k=1}^n b_k \pmod{\Gamma}$ . Then

$$f(z) := \theta \frac{\sigma(z - a_1) \cdots \sigma(z - a_n)}{\sigma(z - b_1) \cdots \sigma(z - b_n)} \quad (\theta \in \mathbf{C} \setminus \{0\})$$

is a meromorphic function on  $\mathbf{C}/\Gamma$  such that  $\{a_1, \dots, a_n\}$  (resp.  $\{b_1, \dots, b_n\}$ ) are the set of zeroes (resp. poles), i.e. the divisor of  $f$  is  $a_1 + \dots + a_n - b_1 - \dots - b_n$ . Conversely, any elliptic function on  $\mathbf{C}/\Gamma$  with the same divisor is of this form.

The meromorphic function  $q(z)$  should have poles of order 1 at  $z = 0, (v_1 + v_2)/2$  (ends) and zeroes of order 1 at  $z = v_1/2, v_2/2$  (umbilic points). Thus  $Q(z)$  can be written as in (4.6).  $\square$

**The case  $(\gamma, n) = (1, 1)$ .** By (4.1) and (4.2), we have two possible cases:

$$(\mu_1^\#, d_1) = (0, -4) \quad \text{or} \quad (1, -3).$$

The second of these cases (the **I**(−3) case) is still unknown, but for the first case **I**(−4), the following proposition provides examples, proven (in Section 5) by deforming from a complete minimal surface in  $\mathbf{R}^3$  of genus 1 with one end satisfying  $d_1 = -4$ .

**Proposition 4.2.** *By deforming the Chen-Gackstatter surface in  $\mathbf{R}^3$  [2], one obtains a one-parameter family of CMC-1 surfaces of type **I**(−4) with dual total absolute curvature  $8\pi$ .*

**The case  $(\gamma, n) = (0, 3)$ .** Here, (4.1) and (2.7) imply  $\mu_j^\# - d_j = 2$  for  $j = 1, 2, 3$ . Moreover, (2.5) implies  $d_1 + d_2 + d_3 \leq -4$ . So (4.2) implies that the possibilities are:

Type **O**(−2, −2, −2) :  $(d_1, d_2, d_3) = (-2, -2, -2)$  and  $(\mu_1^\#, \mu_2^\#, \mu_3^\#) = (0, 0, 0)$ ,

Type **O**(−1, −2, −2) :  $(d_1, d_2, d_3) = (-1, -2, -2)$  and  $(\mu_1^\#, \mu_2^\#, \mu_3^\#) = (1, 0, 0)$ ,

Type **O**(−1, −1, −2) :  $(d_1, d_2, d_3) = (-1, -1, -2)$  and  $(\mu_1^\#, \mu_2^\#, \mu_3^\#) = (1, 1, 0)$ .

In each case, equality holds in (2.1), so all ends are embedded. Since the genus of the surface is 0, we can set  $\overline{M}_0 = \mathbf{C} \cup \{\infty\}$ .

**Surfaces of type **O**(−2, −2, −2).** Such surfaces have three embedded ends with  $d_j = -2$  ( $j = 1, 2, 3$ ), and the irreducible ones are classified in [20, Theorem 2.6]. So here we consider the reducible case.

We may set  $p_1 = 0, p_2 = 1$  and  $p_3 = \infty$ . By (2.5) and (4.3), there are two distinct umbilic points  $q_1$  and  $q_2$  of order 1. Then the Hopf differential  $Q$  must have simple zeroes at  $q_1$  and  $q_2$  and poles of order 2 at 0, 1 and  $\infty$ . Since all three  $\mu_j^\# = 0$ ,  $q_1$  and  $q_2$  are the only branch points of  $G$ . Also,  $G(q_1), G(q_2)$ , and  $G(\infty)$  are all distinct, because  $q_1$  and  $q_2$  are double points of  $G$  and  $\deg G = 2$ . Then, by (2.9), we can set  $G(q_1) = 0, G(q_2) = \infty$ , and  $G(\infty) = 1$ . Thus  $G$  and  $Q$  are written as

$$(4.7) \quad G = \left( \frac{z - q_1}{z - q_2} \right)^2, \quad Q = \theta \frac{(z - q_1)(z - q_2)}{z^2(z - 1)^2} dz^2 \quad (\theta \in \mathbf{C} \setminus \{0\}).$$

*Example 4.3* ( $\mathcal{H}^1$ -reducible examples of type **O**(−2, −2, −2)). For  $s \in \mathbf{R}$  such that

$$(4.8) \quad -4 \frac{1 + 4s + s^2}{1 + 10s + s^2} \in \mathbf{R} \setminus \mathbf{Z},$$

let

$$(4.9) \quad q_1 = \frac{1 + 10s + s^2}{4s(1-s)}, \quad q_2 = \frac{1 + 10s + s^2}{4(s-1)}, \quad \text{and} \quad \theta = -\frac{3}{4q_1q_2}.$$

Consider (E.0) for  $r(z) dz^2 = (S(G)/2) + Q$ , with  $G$  and  $Q$  determined by (4.7) and (4.9). Then the roots of the indicial equation of (E.0) at  $z = 0$  are  $-1/2$  and  $3/2$ , so their difference is  $2 \in \mathbf{Z}$ , and one can check by (A.15) that the log-term coefficient vanishes. Moreover, the difference of the roots of the indicial equation at  $z = 1$  equals the value in (4.8). Hence, by (2) of Proposition 2.2, there exists an  $\mathcal{H}^1$ -reducible CMC-1 immersion  $f: \mathbf{C} \setminus \{0, 1\} \rightarrow H^3$  with  $G$  and  $Q$  as in (4.7) and (4.9). Since each surface is  $\mathcal{H}^1$ -reducible (this follows from the fact that the difference of the roots of the indicial equation is an integer at  $z = 0$  and not an integer at  $z = 1$ ), there exists a one-parameter family of CMC-1 surfaces for each  $s$ , with this  $G$  and  $Q$ . Thus, we have found a 2-parameter family of  $\mathcal{H}^1$ -reducible CMC-1 surfaces of type  $\mathbf{O}(-2, -2, -2)$ .

*Example 4.4* ( $\mathcal{H}^3$ -reducible examples of type  $\mathbf{O}(-2, -2, -2)$ ). For  $m \geq 2$ ,  $m \in \mathbf{Z}$ , let

$$q_1 = \frac{1}{2} \left( 1 + \frac{1}{\sqrt{m}} \right), \quad q_2 = \frac{1}{2} \left( 1 - \frac{1}{\sqrt{m}} \right), \quad \text{and} \quad \theta = -m(m+1).$$

Then a meromorphic function  $g$  on  $\mathbf{C} \cup \{\infty\}$  such that

$$dg = z^{m-1}(z-1)^{m-1}(z-q_1)(z-q_2) dz$$

satisfies equation (2.10) for  $G$  and  $Q$  as in (4.7). Since  $g$  is meromorphic,  $\rho_g(\tau)$  is the identity for all  $\tau \in \pi_1(\mathbf{C} \setminus \{0, 1\})$ , so Lemma 2.1 implies there exists an  $\mathcal{H}^3$ -reducible CMC-1 immersion  $f: \mathbf{C} \setminus \{0, 1\} \rightarrow H^3$  whose hyperbolic Gauss map, Hopf differential, and secondary Gauss map are  $G$ ,  $Q$ , and  $g$ , respectively.

**Surfaces of type  $\mathbf{O}(-1, -2, -2)$ .** In this case, we will see that there is a 2-parameter family of  $\mathcal{H}^1$ -reducible surfaces, and countably many  $\mathcal{H}^3$ -reducible families. By (2.5), there exists one umbilic point of order 1. Without loss of generality, we can set the ends to be  $(p_1, p_2, p_3) = (0, 1, p)$  ( $p \in \mathbf{C} \setminus \{0, 1\}$ ) and the umbilic point to be  $q_1 = \infty$ . Then the Hopf differential  $Q$  has a pole of order 2 (resp. order 1) at  $z = 1, p$  (resp.  $z = 0$ ) and has no zeroes on  $\mathbf{C}$ , so it has the form

$$Q = \frac{\theta dz^2}{z(z-1)^2(z-p)^2} \quad (\theta \in \mathbf{C} \setminus \{0\}).$$

By (2.3) and the fact  $\mu_1^\# = 1$ ,  $G$  has branch points of order 1 at  $z = 0$  and  $\infty$ . Then, by (2.9), we may assume  $G = z^2$ , because  $\deg G = 2$ . Consider the ordinary differential equation (E.0) with  $r(z) dz^2 = (S(G)/2) + Q$ . At the singularity  $z = 0$ ,  $r(z)$  expands as

$$r(z) = -\frac{3}{4} \frac{1}{z^2} + \frac{\theta}{p^2} \frac{1}{z} + \frac{2\theta(p+1)}{p^3} + O(z).$$

Thus the difference of the roots of the indicial equation of (E.0) at  $z = 0$  is 2. Then, by (A.15), the log-term coefficient of (E.0) at  $z = 0$  vanishes if and only if  $\theta = -2p(p+1)$ . Hence, if such a surface exists,  $G$  and  $Q$  are

$$(4.10) \quad G = z^2, \quad Q = \frac{-2p(p+1)}{z(z-1)^2(z-p)^2} dz^2 \quad (p \in \mathbf{C} \setminus \{0, 1\}).$$

For  $G$  and  $Q$  as in (4.10),  $r(z)$  expands at the singularity  $z = 1$  as

$$r(z) = \frac{-2p(p+1)}{(1-p)^2} \frac{1}{(z-1)^2} + O((z-1)^{-1}).$$

Then the roots of the indicial equation of (E.0) at  $z = 1$  are

$$\lambda_1 = 2 + \frac{2}{p-1}, \quad \lambda_2 = -1 - \frac{2}{p-1}.$$

So  $\lambda_1 - \lambda_2 \in \mathbf{Z}$  exactly when  $4/(p-1) \in \mathbf{Z}$ . Then, by Proposition 2.2, we have

**Theorem 4.5.** *Let  $p \in \mathbf{R}$  such that  $p \neq 1$  and  $4/(p-1) \notin \mathbf{Z}$ . Then there exists a conformal  $\mathcal{H}^1$ -reducible CMC-1 immersion  $f: M = \mathbf{C} \cup \{\infty\} \setminus \{0, 1, p\}$  with  $\text{TA}(f^\#) = 8\pi$  and hyperbolic Gauss map and Hopf differential as in (4.10). Moreover, all  $\mathcal{H}^1$ -reducible surfaces with  $\text{TA}(f^\#) = 8\pi$  of type  $\mathbf{O}(-1, -2, -2)$  are given in this manner.*

The above discussion yields that all CMC-1 surfaces of type  $\mathbf{O}(-1, -2, -2)$  are reducible. So it only remains to classify the  $\mathcal{H}^3$ -reducible case:

**Theorem 4.6.** *Let  $r \geq 3$  be an integer and  $p = (r+2)/(r-2)$ . Then there exists a conformal  $\mathcal{H}^3$ -reducible CMC-1 immersion  $f: M = \mathbf{C} \cup \{\infty\} \setminus \{0, 1, p\}$  with  $\text{TA}(f^\#) = 8\pi$  whose hyperbolic Gauss map and Hopf differential are as in (4.10). Moreover, all  $\mathcal{H}^3$ -reducible surfaces with  $\text{TA}(f^\#) = 8\pi$  of type  $\mathbf{O}(-1, -2, -2)$  are given in this manner.*

*Proof.* For given  $r \geq 3$ , there is a meromorphic function  $g$  on  $\mathbf{C} \cup \{\infty\}$  so that

$$(4.11) \quad dg = \frac{z(z-p)^{r-2}}{(z-1)^{r+2}} dz,$$

since the right-hand side of (4.11) has no residue. One can check that  $S(g) - S(G) = 2Q$  when  $p = (r+2)/(r-2)$ . Hence, by Lemma 2.1, there exists an  $\mathcal{H}^3$ -reducible CMC-1 immersion  $f: \mathbf{C} \cup \{\infty\} \setminus \{0, 1, p\} \rightarrow H^3$  with  $G$  and  $Q$  as in (4.10) and secondary Gauss map  $g$  satisfying (4.11).

Conversely, let  $f: \mathbf{C} \cup \{\infty\} \setminus \{0, 1, p\} \rightarrow H^3$  be an  $\mathcal{H}^3$ -reducible CMC-1 immersion of type  $\mathbf{O}(-1, -2, -2)$  with  $\text{TA}(f^\#) = 8\pi$ . Then  $G$  and  $Q$  are as in (4.10). Let  $m_2$  (resp.  $m_3$ ) be the difference of the roots of the indicial equation of (E.0) at  $z = 1$  (resp.  $z = p$ ) for such  $G$  and  $Q$ . Then we have  $m_2 = |3 + (4/(p-1))|$  and  $m_3 = |1 + (4/(p-1))|$ . Since  $f$  is  $\mathcal{H}^3$ -reducible,  $m_2$  and  $m_3$  are positive integers (so also  $4/(p-1) \in \mathbf{Z}$ ). We may assume  $m_2 \geq m_3$ . (If not, we can exchange the two ends  $p$  and  $1$ , by changing  $p$  and  $z$  to  $1/p$  and  $z/p$ . Using (2.9), we see that (4.10) is unchanged.)

Suppose that  $m_2 = m_3 = 1$ , then  $g$  is not branched at both  $1$  and  $p$ . Noting that the branching orders of  $g$  and  $G$  are equal at any finite point of the surface (this follows from equation (2.10)), we see that  $g$  has branch points of order 1 at  $0$  and  $\infty$  and no other branch points. So  $g$  has degree 2 and  $g = a \star z^2$  for some  $a \in \text{SL}(2, \mathbf{C})$  and so  $Q = (1/2)(S(g) - S(G)) = 0$ , which is impossible.

Thus  $m_2 \geq 2$ , and it follows that  $4/(p-1)$  is a positive integer. By setting  $r = 2 + (4/(p-1)) \geq 3$ , we have

$$m_2 = 3 + \frac{4}{p-1} = r+1, \quad m_3 = 1 + \frac{4}{p-1} = r-1, \quad \text{and} \quad p = \frac{r+2}{r-2}.$$

Thus  $G$  and  $Q$  are as in (4.10) with  $p = (r+2)/(r-2)$ .  $\square$

**Surfaces of type  $\mathbf{O}(-1, -1, -2)$ .** In this case, by (2.5), the surface has no umbilic points. We set the ends  $(p_1, p_2, p_3) = (0, 1, \infty)$ . The Hopf differential is then

$$(4.12) \quad Q = \frac{\theta dz^2}{z(z-1)}, \quad (\theta \in \mathbf{C} \setminus \{0\}).$$

The hyperbolic Gauss map  $G$  is a meromorphic function on  $\mathbf{C} \cup \{\infty\}$  of degree 2 with branch points of order 1 at  $z = 0$  and  $z = 1$ . Hence we may set

$$(4.13) \quad G = \left( \frac{z-1}{z} \right)^2.$$

**Theorem 4.7.** *Any complete CMC-1 immersion that is of type  $\mathbf{O}(-1, -1, -2)$  with  $\text{TA}(f^\#) = 8\pi$  is congruent to an  $\mathcal{H}^3$ -reducible CMC-1 immersion  $f: M = \mathbf{C} \setminus \{0, 1\} \rightarrow H^3$  with hyperbolic Gauss map and Hopf differential*

$$G = \left( \frac{z-1}{z} \right)^2, \quad Q = \frac{-2 dz^2}{z(z-1)}.$$

*Proof.* Consider equation (E.0) for  $G$  and  $Q$  in (4.13) and (4.12) respectively. Then the roots of the indicial equations of (E.0) are  $-1/2$  and  $3/2$  at both  $z = 0$  and  $z = 1$ . By (A.15), the log-term coefficients at  $z = 0$  and at  $z = 1$  both vanish if and only if  $\theta = -2$ . By Proposition 2.2, the corresponding 3-parameter family of CMC-1 immersions consists of immersions that are all well-defined on  $M = \mathbf{C} \setminus \{0, 1\}$  and are  $\mathcal{H}^3$ -reducible.  $\square$

**The case  $(\gamma, n) = (0, 2)$ .** In this case, (4.1) and (2.7) imply that

$$(\mu_1^\# - d_1, \mu_2^\# - d_2) = (2, 4) \quad \text{or} \quad (\mu_1^\# - d_1, \mu_2^\# - d_2) = (3, 3).$$

Then, by (4.2), all possibilities are:

- Type  $\mathbf{O}(-2, -4)$  :  $(d_1, d_2) = (-2, -4)$  and  $(\mu_1^\#, \mu_2^\#) = (0, 0)$ ,
- Type  $\mathbf{O}(-2, -3)$  :  $(d_1, d_2) = (-2, -3)$  and  $(\mu_1^\#, \mu_2^\#) = (0, 1)$  or  $(1, 0)$ ,
- Type  $\mathbf{O}(-1, -4)$  :  $(d_1, d_2) = (-1, -4)$  and  $(\mu_1^\#, \mu_2^\#) = (1, 0)$ ,
- Type  $\mathbf{O}(-1, -3)$  :  $(d_1, d_2) = (-1, -3)$  and  $(\mu_1^\#, \mu_2^\#) = (1, 1)$ ,
- Type  $\mathbf{O}(-3, -3)$  :  $(d_1, d_2) = (-1, -3)$  and  $(\mu_1^\#, \mu_2^\#) = (0, 0)$ ,
- Type  $\mathbf{O}(-2, -2)$  :  $(d_1, d_2) = (-2, -2)$  and  $(\mu_1^\#, \mu_2^\#) = (1, 1)$ .

Since the surface has genus 0, we can set  $\overline{M}_0 = \mathbf{C} \cup \{\infty\}$  and  $M = \mathbf{C} \cup \{\infty\} \setminus \{p_1, p_2\}$ . Since  $\pi_1(M)$  is commutative, all surfaces of these types are reducible.

**Surfaces of type  $\mathbf{O}(-3, -3)$ .** There exists a minimal surface in  $\mathbf{R}^3$  of class  $\mathbf{O}(-3, -3)$  with total absolute curvature  $8\pi$  [6]. The following is proven in Section 5:

**Proposition 4.8.** *By deforming the minimal surface of type  $\mathbf{O}(-3, -3)$  in  $\mathbf{R}^3$ , one obtains a one-parameter family of CMC-1 surfaces of type  $\mathbf{O}(-3, -3)$  with dual total absolute curvature  $8\pi$ .*

**Surfaces of type  $\mathbf{O}(-2, -4)$ .** In this case, by (2.5) and (4.3), such a surface has two distinct umbilic points of order 1. We may set the ends to be  $(p_1, p_2) = (0, \infty)$  and the umbilic points to be  $(q_1, q_2) = (1, q)$ ,  $q \in \mathbf{C} \setminus \{0, 1\}$ , on  $\mathbf{C} \cup \{\infty\}$ . Then we may assume

$$(4.14) \quad G = \left( \frac{z-q}{z-1} \right)^2, \quad Q = \frac{\theta(z-1)(z-q)}{z^2} dz^2 \quad (\theta \in \mathbf{C} \setminus \{0\}).$$

For such  $G$  and  $Q$ , the roots of the indicial equation of (E.0) at  $z = 0$  are

$$\lambda_1 = \frac{1}{2} \left( 1 + \sqrt{1 - 4\theta q} \right), \quad \lambda_2 = \frac{1}{2} \left( 1 - \sqrt{1 - 4\theta q} \right).$$

Then, by (2) of Proposition 2.2, we have

**Theorem 4.9.** *Let  $\theta \in \mathbf{C} \setminus \{0\}$  and  $q \in \mathbf{C} \setminus \{0, 1\}$  be complex numbers such that*

$$\sqrt{1 - 4\theta q} \in \mathbf{R} \setminus \mathbf{Z}.$$

*Then there exists a conformal  $\mathcal{H}^1$ -reducible CMC-1 immersion  $f: \mathbf{C} \setminus \{0\} \rightarrow H^3$  of type  $\mathbf{O}(-2, -4)$  with  $\text{TA}(f^\#) = 8\pi$  whose hyperbolic Gauss map and Hopf differential are as in (4.14). Moreover, all  $\mathcal{H}^1$ -reducible surfaces with  $\text{TA}(f^\#) = 8\pi$  of type  $\mathbf{O}(-2, -4)$  are given in this manner.*

It only remains to consider the  $\mathcal{H}^3$ -reducible case:

**Theorem 4.10.** *Let  $s \in \mathbf{R}$  such that  $\sqrt{1 - 4s} \geq 2$  is an integer. Then there exists at least 1 and at most  $\sqrt{1 - 4s}$  conformal  $\mathcal{H}^3$ -reducible CMC-1 immersions  $f: \mathbf{C} \setminus \{0\} \rightarrow H^3$  of type  $\mathbf{O}(-2, -4)$  with  $\text{TA}(f^\#) = 8\pi$  whose hyperbolic Gauss map and Hopf differential are as in (4.14). Moreover, all  $\mathcal{H}^3$ -reducible surfaces with  $\text{TA}(f^\#) = 8\pi$  of type  $\mathbf{O}(-2, -4)$  are given in this manner.*

*Proof.* For  $G$  and  $Q$  in (4.14), equation (E.1) $^\#$  becomes

$$(4.15) \quad z^2 X'' + z \left\{ 2 + \frac{4z}{1-z} \right\} X' + \{\theta(z-1)(z-q)\} X = 0.$$

By Lemma 2.3 and Proposition 2.2, it is enough to show that there exists data  $(G, Q)$  such that the difference of the roots of the indicial equation of (4.15) at  $z = 0$  is an integer and the log-term vanishes.

The coefficients of (4.15) expand as

$$z \left\{ 2 + \frac{4z}{1-z} \right\} = z \left\{ 2 + 4 \sum_{j=1}^{\infty} z^j \right\} \quad \text{and} \quad \theta(z-1)(z-q) = \theta q - \theta(1+q)z + \theta z^2$$

for  $z$  sufficiently close to 0. Assume the roots  $\lambda_1, \lambda_2$  of the indicial equation of (4.15) satisfy  $\lambda_1 - \lambda_2 = m \in \mathbf{Z}^+$ . Then

$$(4.16) \quad s := \theta q = \frac{1 - m^2}{4} \quad \text{and} \quad \lambda_2 = -\frac{m+1}{2} \quad (m \geq 2).$$

Let

$$\mu_j = \begin{cases} \frac{1}{j(m-j)} & (j = 1, 2, \dots, m-1) \\ -\frac{1}{m} & (j = m) \end{cases}.$$

Then by Proposition A.3 in Appendix A, the log-term coefficient  $c$  of (4.15) is given by  $c = a_m$ , where  $a_0 = 1$  and

$$(4.17) \quad a_j = \mu_j \left[ \left( \sum_{k=0}^{j-2} (4k - 2m - 2)a_k \right) + \theta a_{j-2} + \left( \frac{1}{4}(m+1)(m-9) - 4 + 4j - \theta \right) a_{j-1} \right]$$

for  $j = 1, \dots, m$ . Hence  $a_j$  is a polynomial in  $\theta$  of order  $j$ . We now define  $t_0 = 1$ , and we define  $t_j$  and  $u_j$  for  $j = 1, \dots, m$  by the relations

$$a_j = t_j \theta^j + u_j \theta^{j-1} + \dots \quad (j = 1, 2, \dots, m).$$

It follows that  $t_j = -\mu_j t_{j-1}$ , and hence  $t_m \neq 0$ . Then, defining  $\Lambda_j := u_j/t_j$ , we also have

$$\Lambda_j = \Lambda_{j-1} - \frac{(m+1)(m-9)}{4} - 4j + 4 + (j-1)(m-j+1)$$

for  $j = 2, \dots, m$ . Since  $\Lambda_1 = -(m+1)(m-9)/4$ , we have

$$\Lambda_m = \sum_{j=2}^m \left[ -\frac{(m+1)(m-9)}{4} - 4j + 4 + (j-1)(m-j+1) \right] = \frac{m}{12}(49 - m^2).$$

If the only roots of the polynomial

$$c = t_m \theta^m + u_m \theta^{m-1} + \dots = t_m (\theta^m + \Lambda_m \theta^{m-1} + \dots) = 0$$

with respect to  $\theta$  are 0 and  $(1 - m^2)/4 < 0$ , then it follows that  $\Lambda_m$  would be nonnegative. However,  $\Lambda_m < 0$  for all  $m \geq 8$ , hence this polynomial must have some root  $\theta \in \mathbf{C} \setminus \{0, (1 - m^2)/4\}$ , and then  $q = (1 - m^2)/(4\theta) \in \mathbf{C} \setminus \{0, 1\}$ . For this  $\theta$  and  $q$ , we have  $c = 0$ , and thus we have at least one surface for each  $m \geq 8$ . Since  $c$  is a polynomial of degree  $m$  in  $\theta$ , there are at most  $m$  roots, and hence at most  $m$  surfaces.

For  $m \leq 7$ , one can check by explicitly computing the polynomial for  $c$  that there is always at least one root  $\theta \in \mathbf{C} \setminus \{0, (1 - m^2)/4\}$ .  $\square$

**Surfaces of type  $\mathbf{O}(-2, -3)$  with  $\mu_1^\# = \mathbf{0}$ .** Here, by (2.5), there exists only one umbilic point of order 1. We set the ends to be  $(p_1, p_2) = (1, \infty)$  and the umbilic point to be  $q_1 = 0$ . We may assume

$$(4.18) \quad G = z^2, \quad Q = \frac{\theta z dz^2}{(z-1)^2} \quad (\theta \in \mathbf{C} \setminus \{0\}).$$

Then the roots of the indicial equation of (E.0) at  $z = 1$  are

$$\lambda_1 = \frac{1}{2} \left( 1 + \sqrt{1 - 4\theta} \right), \quad \lambda_2 = \frac{1}{2} \left( 1 - \sqrt{1 - 4\theta} \right).$$

Hence, by Proposition 2.2, we have

**Theorem 4.11.** *Let  $\theta \in \mathbf{R}$  such that  $\sqrt{1 - 4\theta} \in \mathbf{R} \setminus \mathbf{Z}$ . Then there exists a conformal  $\mathcal{H}^1$ -reducible CMC-1 immersion  $f: \mathbf{C} \setminus \{1, \infty\} \rightarrow H^3$  of type  $\mathbf{O}(-2, -3)$  with  $\text{TA}(f^\#) = 8\pi$  whose hyperbolic Gauss map and Hopf differential are as in (4.18). Moreover, all  $\mathcal{H}^1$ -reducible surfaces of type  $\mathbf{O}(-2, -3)$  with  $(\mu_1^\#, \mu_2^\#) = (0, 1)$  and  $\text{TA}(f^\#) = 8\pi$  are given in this manner.*

Now we will show that there are no  $\mathcal{H}^3$ -reducible surfaces of this type, by showing that the log-term coefficient at  $z = 1$  of  $((E.1)^\#)$  cannot be zero. With the data as in (4.18), equation  $(E.1)^\#$  becomes

$$(z-1)^2 X'' + 2(z-1)X' + \theta(1+(z-1))X = 0,$$

and so  $p_0 = 2$ ,  $q_0 = q_1 = \theta$ ,  $p_j = 0$  for  $j \geq 1$ , and  $q_j = 0$  for  $j \geq 2$ , where the  $p_j$  and  $q_j$  are as defined in (A.3). Therefore, by Proposition A.3, we have  $c = -\theta^m / (m!(m-1)!) \neq 0$ .

**Surfaces of type  $\mathbf{O}(-2, -3)$  with  $\mu_1^\# = 1$ .** In this case, we set the ends to be  $(p_1, p_2) = (0, \infty)$  and the only umbilic point to be  $q_1 = 1$ . Then we may assume

$$(4.19) \quad G = \left( \frac{z-1}{z} \right)^2, \quad Q = \frac{\theta(z-1) dz^2}{z^2} \quad (\theta \in \mathbf{C} \setminus \{0\}).$$

Thus the roots of the indicial equation of (E.0) at  $z = 0$  are

$$\lambda_1 = \frac{1}{2} \left( 1 + \sqrt{4+4\theta} \right), \quad \lambda_2 = \frac{1}{2} \left( 1 - \sqrt{4+4\theta} \right).$$

So, by Proposition 2.2, we have

**Theorem 4.12.** *Let  $\theta \in \mathbf{R}$  such that  $\sqrt{4+4\theta} \in \mathbf{R} \setminus \mathbf{Z}$ . Then there exists a conformal  $\mathcal{H}^1$ -reducible CMC-1 immersion  $f: \mathbf{C} \setminus \{0\} \rightarrow H^3$  of type  $\mathbf{O}(-2, -3)$  with  $\mathrm{TA}(f^\#) = 8\pi$  whose hyperbolic Gauss map and Hopf differential are as in (4.19). Moreover, all  $\mathcal{H}^1$ -reducible surfaces of type  $\mathbf{O}(-2, -3)$  with  $(\mu_1^\#, \mu_2^\#) = (1, 0)$  and  $\mathrm{TA}(f^\#) = 8\pi$  are given in this manner.*

Now we will show that there are no  $\mathcal{H}^3$ -reducible surfaces of this type as well, again by showing that a log-term coefficient cannot be zero. With  $G$  and  $Q$  as in (4.19), equation  $(E.2)^\#$  becomes

$$z^2 X'' - zX' + \theta(z-1)X = 0,$$

and so  $p_0 = -1$ ,  $-q_0 = q_1 = \theta$ ,  $p_j = 0$  for  $j \geq 1$ , and  $q_j = 0$  for  $j \geq 2$ , where the  $p_j$  and  $q_j$  are as defined in (A.3). Hence again, by Proposition A.3, we have  $c \neq 0$ .

**Surfaces of type  $\mathbf{O}(-1, -4)$ .** We set the ends to be  $(p_1, p_2) = (0, 1)$  and the single umbilic point to be  $q_1 = \infty$ , then we may assume

$$(4.20) \quad G = z^2, \quad Q = \frac{\theta dz^2}{z(z-1)^4} \quad (\theta \in \mathbf{C} \setminus \{0\}).$$

The roots of the indicial equation of (E.0) for such  $G$  and  $Q$  at  $z = 0$  are  $3/2$  and  $-1/2$ . Then, by Lemma A.15, the log-term coefficient at  $z = 0$  vanishes if and only if  $\theta = -4$ . Thus

**Theorem 4.13.** *Any complete CMC-1 immersion of type  $\mathbf{O}(-1, -4)$  with  $\mathrm{TA}(f^\#) = 8\pi$  is congruent to an  $\mathcal{H}^3$ -reducible CMC-1 immersion  $f: M = \mathbf{C} \cup \{\infty\} \setminus \{0, 1\} \rightarrow H^3$  with hyperbolic Gauss map and Hopf differential*

$$G = z^2, \quad Q = \frac{-4 dz^2}{z(z-1)^4}.$$

**Surfaces of type  $\mathbf{O}(-1, -3)$ .** In this case, there are no umbilic points, by (2.5). Then, if we set the ends to be  $(p_1, p_2) = (0, \infty)$ , we may assume

$$G = z^2, \quad Q = \frac{\theta}{z} dz^2 \quad (\theta \in \mathbf{C} \setminus \{0\}).$$

The roots of the indicial equation of (E.0) at  $z = 0$  are  $3/2$  and  $-1/2$ , and the log-term coefficient vanishes if and only if  $\theta = 0$ , by (A.15). So this case is impossible.

**Surfaces of type  $\mathbf{O}(-2, -2)$ .** Here again there are no umbilic points, by (2.5). If we set the ends to be  $(p_1, p_2) = (0, \infty)$ , we may assume

$$G = z^2, \quad Q = \frac{\theta}{z^2} dz^2 \quad (\theta \in \mathbf{C} \setminus \{0\}).$$

Then the solution  $g$  of the equation  $S(g) - S(G) = 2Q$  is

$$g = az^\mu + b, \quad a \in \mathbf{C} \setminus \{0\}, \quad b \in \mathbf{C} \quad \text{and} \quad \mu = \sqrt{1 - 4\theta}.$$

Hence the function  $g$  satisfies  $\rho_g(\tau) \in \mathrm{SU}(2)$  for all  $\tau \in \pi_1(\mathbf{C} \setminus \{0\})$  if and only if  $\mu \in \mathbf{Z}$ , or  $\mu \in \mathbf{R}$  and  $b = 0$ . In the second case, the surface is a double cover of a catenoid cousin. The first case is a warped catenoid cousin with  $m = 2$  in Theorem 6.2 of [15] (see also [13]).

**The case  $(\gamma, n) = (0, 1)$ .** In this case, we can set  $M = \mathbf{C}$ . Since  $M$  is simply connected, we have no period problem. By (4.1) and (4.2),  $d_1 = -5$  or  $-6$ .

For the case of  $\mathbf{O}(-5)$ , there is one umbilic point, which we may suppose is at  $q_1 = 0$ . By (4.1), we have  $\mu_1^\# = 1$ , so we may assume

$$(4.21) \quad G = z^2, \quad Q = \theta z dz^2 \quad (\theta \in \mathbf{C} \setminus \{0\}).$$

For the case of  $\mathbf{O}(-6)$ , there are two umbilic points of order 1. Without loss of generality, we can set them to be  $(q_1, q_2) = (0, 1)$ . So, since  $\mu_1^\# = 0$ , we may assume

$$(4.22) \quad G = \left(\frac{z-1}{z}\right)^2, \quad Q = \theta z(z-1) dz^2 \quad (\theta \in \mathbf{C} \setminus \{0\}).$$

**Theorem 4.14.** *A CMC-1 surface of genus zero with one end such that  $\mathrm{TA}(f^\#) = 8\pi$  is congruent to an immersion  $f: \mathbf{C} \rightarrow H^3$  with hyperbolic Gauss map and Hopf differential as in (4.21) or (4.22). Moreover, such a surface is  $\mathcal{H}^3$ -reducible.*

## 5. DEFORMATION OF MINIMAL SURFACES TO CMC-1 SURFACES

In this section, we prove Propositions 4.2 and 4.8. For this, we will need a method from [10] that produces a 1-parameter family of CMC-1 surfaces in  $H^3$  from a corresponding minimal surface in  $\mathbf{R}^3$ , so we describe that method first.

We start with a complete minimal surface  $f_0: M \rightarrow \mathbf{R}^3$  of finite total curvature. We require the immersion to be symmetric in the following sense, a condition that generically eliminates virtually all minimal surfaces, but eliminates none of the better known surfaces, which all have symmetries:

**Symmetry condition:** There is a disk  $D \subset M$  so that  $f_0(D)$  is bounded by non-straight planar geodesics.

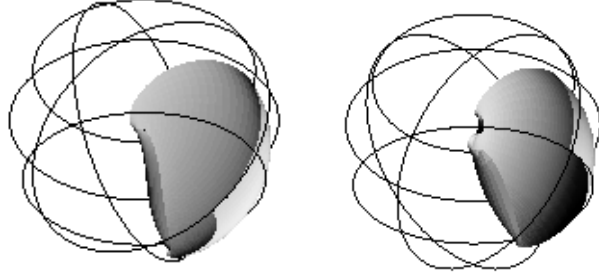


FIGURE 2. Genus 0 and genus 1 Enneper cousin duals. Each surface has a single end that triply wraps around its limiting point at the south pole of the sphere at infinity. These surfaces are of type  $\mathbf{O}(-4)$  and  $\mathbf{I}(-4)$ , and have  $\mathrm{TA}(f^\#) = 4\pi$  and  $\mathrm{TA}(f^\#) = 8\pi$ . In both cases only one of four congruent pieces (bounded by planar geodesics) of the surface is shown.

If  $f_0$  is symmetric with respect to a disk  $D$ , then  $f_0(D)$  generates the full surface by reflections across planes containing the boundary planar geodesics of  $\partial f_0(D)$ , by the Schwarz reflection principle [8]. Since the surface has finite total curvature, it is shown in [10] that the boundary  $\partial f_0(D)$  is contained entirely in only either one plane  $P_1$ , or in two intersecting planes  $P_1, P_2$ , or in three planes  $P_1, P_2$ , and  $P_3$  in general position. Let the boundary planar geodesics of  $f_0(D)$  contained in  $P_j$  be called  $S_{j,1}, S_{j,2}, \dots, S_{j,\delta_j}$  ( $j = 1, \dots, s$ , for  $s = 1, 2$  or  $3$ ).

We now define non-degeneracy of the period problems. Let  $\delta$  be the number of  $S_{j,l}$  minus the number of planes ( $\delta = \delta_1 + \delta_2 + \delta_3 - 3$  if  $s = 3$ ,  $\delta = \delta_1 + \delta_2 - 2$  if  $s = 2$ , and  $\delta = \delta_1 - 1$  if  $s = 1$ ).

**Nondegeneracy condition:** There exists a continuous  $\delta$ -parameter family of minimal disks  $f_{0,\nu}(D)$  (where  $\nu$  lies in a small neighborhood of the origin  $\vec{0} \in \mathbf{R}^\delta$ ) such that

- (1)  $f_{0,\vec{0}}(D) = f_0(D)$ .
- (2)  $\partial f_{0,\nu}(D) = \cup_{j=1}^s (\cup_{l=1}^{\delta_j} S_{j,l}(\nu))$  holds, and each  $S_{j,l}(\nu)$  is a planar geodesic lying in a plane  $P_{j,l}(\nu)$  parallel to  $P_j$ .
- (3) Letting  $\mathrm{Per}_{j,l}(\nu)$  ( $j = 1, \dots, s$ ,  $l = 2, \dots, \delta_j$ ) be the oriented distance between the plane  $P_{j,l}(\nu)$  and  $P_{j,1}(\nu)$ , the map from  $\nu$  in  $\mathbf{R}^\delta$  to  $(\mathrm{Per}_{j,l}(\nu))$  in  $\mathbf{R}^\delta$  is an open map onto a small neighborhood of  $\vec{0} \in \mathbf{R}^\delta$ .

**Theorem 5.1** ([10]). *If the minimal immersion  $f_0$  is symmetric and nondegenerate, then there exists a one-parameter family of CMC-1 surfaces in  $H^3$ , each of whose hyperbolic Gauss map and Hopf differential coincide with the Gauss map and Hopf differential of  $f_{0,\nu}(D)$  for some  $\nu \in \mathbf{R}^\delta$ .*

We now consider two applications of this theorem:

**Existence of surfaces of type  $\mathbf{I}(-4)$  with  $\mathrm{TA}(f^\#) = 8\pi$ .** We construct a deformation of the Chen-Gackstatter minimal surface defined on the elliptic curves

$$\overline{M}_1(\nu_1) = \{(z, w) \in (\mathbf{C} \cup \{\infty\})^2 \mid w^2 = z(z-1)(z+\nu_1)\} \quad (\nu_1 \in \mathbf{R}^+),$$

with the point  $p_1$  corresponding to  $z = \infty$  removed ( $p_1$  will be the single end of the surfaces). Let

$$g = \frac{\nu_2 w}{z}, \quad \omega = \frac{z dz}{w} \quad (\nu_2 \in \mathbf{R}^+).$$

We choose the fundamental pieces of the surfaces to be the images under the Weierstrass representation

$$(5.1) \quad \operatorname{Re} \int_{z_0=0}^z ((1 - g^2, i(1 + g^2), 2g) \omega$$

of the half sheets

$$\begin{aligned} \{ (z, w_1 w_2 w_3) \in \overline{M}_1(\nu_1) \mid z \in \mathbf{C}, \operatorname{Im}(z) \geq 0, w_1^2 = z, \\ w_2^2 = z - 1, w_3^2 = z + \nu_1, \arg(w_j) \in [0, \pi), j = 1, 2, 3 \} . \end{aligned}$$

The fundamental pieces are bounded by four planar geodesics, two of which lie in planes parallel to the  $x_1 x_3$ -plane and two of which lie in planes parallel to the  $x_2 x_3$ -plane. Thus  $\delta = 2$ . Note that the period problem is solved, and the Chen-Gackstatter surface is produced, if  $\nu_1 = 1$  and  $\nu_2 = \sqrt{B}$ , where

$$B := \left( \int_0^1 \frac{x dx}{\sqrt{x(1-x^2)}} \right) / \left( \int_0^1 \frac{(1-x^2) dx}{\sqrt{x(1-x^2)}} \right).$$

The oriented distance functions (between the two pairs of parallel planes containing boundary curves of the fundamental pieces) are given by

$$\begin{aligned} \operatorname{Per}_1(\nu_1, \nu_2) &= \int_0^1 (1 - \nu_2^2 x^{-1} (1-x)(x + \nu_1)) \frac{\sqrt{x} dx}{\sqrt{(1-x)(x + \nu_1)}}, \\ \operatorname{Per}_2(\nu_1, \nu_2) &= \int_0^1 \left( 1 - \nu_1 \nu_2^2 x^{-1} (1-x)(x + \frac{1}{\nu_1}) \right) \frac{\sqrt{\nu_1} \sqrt{x} dx}{\sqrt{(1-x)(x + (1/\nu_1))}}. \end{aligned}$$

To see that the period problem is nondegenerate, it is sufficient to check that the Jacobian matrix  $(\partial(\operatorname{Per}_1, \operatorname{Per}_2)/\partial(\nu_1, \nu_2))$  has nonzero determinant at  $(\nu_1, \nu_2) = (1, \sqrt{B})$ . It is easy to check that  $|\partial \operatorname{Per}_1 / \partial \nu_2| = |\partial \operatorname{Per}_2 / \partial \nu_2| \neq 0$  at  $(\nu_1, \nu_2) = (1, \sqrt{B})$ . Since

$$\begin{aligned} \left. \frac{\partial \operatorname{Per}_1}{\partial \nu_1} \right|_{(\nu_1, \nu_2) = (1, \sqrt{B})} &= \int_0^1 \frac{x + B(1-x^2)}{2(x-1)(1+x)^2 \sqrt{x}} \sqrt{1-x^2} dx, \\ \left. \frac{\partial \operatorname{Per}_2}{\partial \nu_1} \right|_{(\nu_1, \nu_2) = (1, \sqrt{B})} &= \int_0^1 \frac{-x(x+2) + B(2+3x)(1-x^2)}{2(x-1)(1+x)^2 \sqrt{x}} \sqrt{1-x^2} dx, \end{aligned}$$

we have

$$\left| \frac{\partial \operatorname{Per}_1}{\partial \nu_1} \right| \neq \left| \frac{\partial \operatorname{Per}_2}{\partial \nu_1} \right|$$

at  $(\nu_1, \nu_2) = (1, \sqrt{B})$ . Thus the determinant of the Jacobian is nonzero, and the period problem is nondegenerate. Hence Theorem 5.1 implies existence of associated CMC-1 surfaces in  $H^3$  of type  $\mathbf{O}(-4)$ . Furthermore, as Theorem 5.1 also implies that the hyperbolic Gauss maps will be  $\nu_2 w/z$ , these surfaces have dual total absolute curvature  $8\pi$ .

**Existence of surfaces of type  $\mathbf{O}(-3, -3)$  with  $\mathbf{TA}(f^\#) = 8\pi$ .** Let  $M = \mathbf{C} \cup \{\infty\} \setminus \{0, \infty\}$  and

$$(5.2) \quad g = \frac{2z^2 + 2az - a^2 - 1}{2(z+1)} + \nu, \quad \text{and} \quad \omega = \frac{(z+1)^2}{z^3} dz,$$

where  $a, \nu \in \mathbf{R}$ .

When  $\nu = 0$ , the Weierstrass representation (5.1) determines a minimal immersion  $f_0: M \rightarrow \mathbf{R}^3$  with finite total curvature of type  $\mathbf{O}(-3, -3)$  ([6, Theorem 4]). For the metric to be nondegenerate at  $z = -1$ , we must assume  $a \neq -1 \pm \sqrt{2}$ .

Since the Hopf differential  $Q = \omega dg$  satisfies  $\overline{Q(\bar{z})} = Q(z)$ , these minimal surfaces each have two planar geodesics that are the images of the positive and negative real axes of  $\mathbf{C}$  under the Weierstrass representation (5.1), and their fundamental pieces are the images of the upper half plane of  $\mathbf{C}$  under (5.1). The two planar geodesics comprise the boundaries of each of the fundamental pieces, and both lie in planes parallel to the  $x_1x_3$ -plane, since  $g$  is real-valued on the real axis. So  $\delta = 1$ , and the oriented distance between the two planes containing the two geodesics is

$$\text{Per}(\nu) := \text{Re} \left( 2\pi i \text{Res}_{z=0} i(1+g^2)\omega \right) = -2\pi\nu(2+2a+\nu),$$

so  $d\text{Per}(\nu)/d\nu$  is nonvanishing at  $\nu = 0$  when  $a \neq -1$ . Thus Theorem 5.1 implies existence of a 1-parameter family of CMC-1 surfaces of type  $\mathbf{O}(-3, -3)$  in  $H^3$  for each  $a \neq -1, -1 \pm \sqrt{2}$  with dual total absolute curvature  $8\pi$  (as  $g$  has degree 2).

#### APPENDIX A.

Here we review some elementary facts in the theory of linear ordinary differential equations. Define a differential operator

$$(A.1) \quad L[u] := z^2 u'' + zp(z)u' + q(z)u \quad \left( ' = \frac{d}{dz} \right).$$

In this note, we shall consider the solution of the ordinary differential equation with a regular singularity at the origin:

$$(A.2) \quad L[u] = 0,$$

where

$$(A.3) \quad p(z) = \sum_{j=0}^{\infty} p_j z^j, \quad q(z) = \sum_{j=0}^{\infty} q_j z^j.$$

It is well-known (and we will see it in this appendix) that (A.2) has two linearly independent solutions  $\{X_1, X_2\}$  of the form

$$X_1 = z^{\lambda_1} \sum_{j=0}^{\infty} \eta_{1,j} z^j, \quad X_2 = \left( z^{\lambda_2} \sum_{j=0}^{\infty} \eta_{2,j} z^j \right) + c X_1 \log z,$$

where  $\eta_{1,0} \neq 0$  and  $\eta_{2,0} \neq 0$ , and where  $\lambda_1$  and  $\lambda_2$  are given by

$$(A.4) \quad \lambda_1 = \frac{1}{2} \{(1-p_0) + m\}, \quad \lambda_2 = \frac{1}{2} \{(1-p_0) - m\}, \quad m = \sqrt{(1-p_0)^2 - 4q_0}.$$

The coefficient  $c$  is called *the log-term coefficient* of differential equation (A.2), which may be nonzero only when  $\lambda_1 - \lambda_2 \in \mathbf{Z}$ .

We shall give a method for computing the coefficient  $c$ . First, we shall describe two linearly independent solutions  $X_1, X_2$  as a formal power series. If we find a solution of (A.2) as a formal power series, a well-known existence theorem from the theory of ordinary differential equations says that it will converge in a sufficiently small neighborhood of the origin [3]. So the formal treatment is sufficient for the computation of  $c$ .

For a complex variable  $\lambda$ , define rational functions  $\zeta_j(\lambda)$  for non-negative integers  $j$  as

$$(A.5) \quad \zeta_0(\lambda) = 1, \quad \text{and} \quad \zeta_j(\lambda) = -\frac{1}{\varphi(\lambda+j)} \sum_{k=0}^{j-1} r_{j,k}(\lambda) \zeta_k(\lambda) \quad (j = 1, 2, \dots),$$

where

$$\varphi(t) = t(t-1) + tp_0 + q_0, \quad r_{j,k}(\lambda) = (\lambda+k)p_{j-k} + q_{j-k},$$

and we set

$$(A.6) \quad X(\lambda) := z^\lambda \sum_{n=0}^{\infty} \zeta_n(\lambda) z^n.$$

Applying the operator  $L$  to  $X(\lambda)$ , we have

$$(A.7) \quad L[X(\lambda)] = z^\lambda \left\{ \varphi(\lambda) + \sum_{j=1}^{\infty} \left( \varphi(\lambda+j) \zeta_j(\lambda) + \sum_{k=0}^{j-1} r_{j,k}(\lambda) \zeta_k(\lambda) \right) z^j \right\} = z^\lambda \varphi(\lambda)$$

The quadratic equation

$$(A.8) \quad \varphi(t) = t(t-1) + tp_0 + q_0 = 0$$

is called the *indicial equation* of the equation (A.2), and we denote the solutions of (A.8) by  $\lambda_1$  and  $\lambda_2$ .

First, we consider the case  $\lambda_1 - \lambda_2 \notin \mathbb{Z}$ . In this case,  $\varphi(\lambda_l + j) \neq 0$  ( $l = 1, 2$ ) for any positive integer  $j$ , and then  $\zeta_j(\lambda_l)$  ( $l = 1, 2$ ) in (A.5) are all well-defined. Moreover, by (A.7),  $X_1 := X(\lambda_1)$  and  $X(\lambda_2)$  are linearly independent solutions of (A.2).

Next, assume  $m := \lambda_1 - \lambda_2$  is a non-negative integer. Since  $\varphi(\lambda_1 + j) \neq 0$  for any positive integer  $j$ ,  $X_1 := X(\lambda_1)$  is a well-defined power series and a solution of (A.2).

**The case  $m = 0$ .** Assume  $\lambda_1 = \lambda_2$ . Since  $\varphi(\lambda_1 + j) \neq 0$  for any positive integer  $j$ ,  $\lambda = \lambda_1$  is not a pole of  $\zeta_j(\lambda)$  for each  $j$ . Hence

$$\zeta_j(\lambda_1) \quad \text{and} \quad \left. \frac{\partial}{\partial \lambda} \right|_{\lambda=\lambda_1} \zeta_j(\lambda) \quad (j = 0, 1, 2, \dots)$$

are well-defined. Let

$$(A.9) \quad X_2 := \left. \frac{\partial}{\partial \lambda} \right|_{\lambda=\lambda_1} X(\lambda) = z^{\lambda_1} \sum_{n=0}^{\infty} \left( \left. \frac{\partial}{\partial \lambda} \right|_{\lambda=\lambda_1} \zeta_n(\lambda) \right) z^n + X_1 \cdot \log z.$$

**Proposition A.1.** *If  $m = \lambda_1 - \lambda_2 = 0$ ,  $X_2$  in (A.9) is a solution of (A.2). Moreover, the log-term coefficient of (A.2) never vanishes.*

*Proof.* It is enough to show that  $X_2$  is a solution of (A.2). In fact, by (A.7),

$$L[X_2] = \left. \frac{\partial}{\partial \lambda} \right|_{\lambda=\lambda_1} L[X(\lambda)] = z^{\lambda_1} \left. \frac{\partial}{\partial \lambda} \right|_{\lambda=\lambda_1} \varphi(\lambda) + z^{\lambda_1} \varphi(\lambda_1) \log z = 0,$$

because  $\varphi(\lambda) = (\lambda - \lambda_1)^2$ . □

**The case  $m > 0$ .** Assume  $m = \lambda_1 - \lambda_2$  is a positive integer. Since  $\varphi(t) = (t - \lambda_2 - m)(t - \lambda_2)$ ,  $\varphi(\lambda_2 + j)$  does not vanish for each positive integer  $j$ , except for  $j = m$ . Then  $\zeta_j(\lambda)$  has no pole at  $\lambda = \lambda_2$  for  $j = 1, 2, \dots, m-1$ , and may have a pole of order one at  $\lambda = \lambda_2$  for  $j \geq m$ . Hence

$$\lim_{\lambda \rightarrow \lambda_2} \{(\lambda - \lambda_2) \zeta_j(\lambda)\} \quad \text{and} \quad \left. \frac{\partial}{\partial \lambda} \right|_{\lambda=\lambda_2} [(\lambda - \lambda_2) \zeta_j(\lambda)]$$

are well-defined. Moreover,

$$(A.10) \quad \lim_{\lambda \rightarrow \lambda_2} \{(\lambda - \lambda_2) \zeta_j(\lambda)\} = 0 \quad (j = 1, 2, \dots, m-1)$$

holds. Let

$$\xi_j := \lim_{\lambda \rightarrow \lambda_2} \{(\lambda - \lambda_2)\zeta_{m+j}(\lambda)\} \quad (j = 0, 1, 2, \dots)$$

and set  $c := \xi_0 = \lim_{\lambda \rightarrow \lambda_2} \{(\lambda - \lambda_2)\zeta_m(\lambda)\}$ . Then by (A.5) and (A.10), we have

$$\xi_0 = c \quad \text{and} \quad \xi_j = \frac{-1}{\varphi(\lambda_2 + m + j)} \sum_{k=0}^{j-1} r_{j,k}(\lambda_2 + m)\xi_k \quad (j = 1, 2, \dots).$$

Comparing this with (A.5), we have  $\xi_j = c\zeta_j(\lambda_1)$  ( $j = 1, 2, \dots$ ), because  $\lambda_1 = \lambda_2 + m$ .

Let

$$(A.11) \quad X_2 := \left. \frac{\partial}{\partial \lambda} \right|_{\lambda=\lambda_2} [(\lambda - \lambda_2)X(\lambda)].$$

Then by (A.10), we have

$$\begin{aligned} X_2 &= z^{\lambda_2} \left( \sum_{j=0}^{\infty} \xi_j z^{j+m} \right) \log z + z^{\lambda_2} \sum_{j=0}^{\infty} \left. \frac{\partial}{\partial \lambda} \right|_{\lambda=\lambda_2} \{(\lambda - \lambda_2)\zeta_j(\lambda)\} z^j \\ &= c \log z X_1 + z^{\lambda_2} \sum_{j=0}^{\infty} \left. \frac{\partial}{\partial \lambda} \right|_{\lambda=\lambda_2} \{(\lambda - \lambda_2)\zeta_j(\lambda)\} z^j. \end{aligned}$$

**Proposition A.2.** *If  $m = \lambda_1 - \lambda_2$ , is a positive integer,  $X_2$  in (A.11) is a solution of (A.2). Moreover, the log-term coefficient  $c$  of (A.2) is given by*

$$(A.12) \quad c := \xi_0 = \lim_{\lambda \rightarrow \lambda_2} \{(\lambda - \lambda_2)\zeta_m(\lambda)\}.$$

*Proof.* By (A.7),

$$L[X_2] = \lim_{\lambda \rightarrow \lambda_2} \frac{\partial}{\partial \lambda} \left( z^\lambda (\lambda - \lambda_2) \varphi(\lambda) \right) = 0,$$

because  $\varphi(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)$ .  $\square$

We have established the following recursive formula for  $c$ , which follows immediately from equation (A.12):

**Proposition A.3.** *If the difference  $m$  of the roots of the indicial equation of (A.2) is a positive integer, then the log-term coefficient  $c$  is*

$$(A.13) \quad c = -\frac{1}{m} \sum_{k=0}^{m-1} ((\lambda_2 + k)p_{m-k} + q_{m-k}) a_k,$$

where  $a_0 = 1$  and

$$a_j = \frac{1}{j(m-j)} \sum_{k=0}^{j-1} ((\lambda_2 + k)p_{j-k} + q_{j-k}) a_k \quad (j = 1, 2, \dots, m-1).$$

*Proof.* Since  $\varphi(t) = (t - \lambda_2)(t - \lambda_2 - m)$ ,  $\varphi(\lambda_2 + j) \neq 0$  for  $j = 1, \dots, m-1$  and then  $a_j = \zeta_j(\lambda_2)$  ( $j = 1, \dots, m-1$ ) is well-defined. Hence, by (A.12),

$$\begin{aligned} c &= \lim_{\lambda \rightarrow \lambda_2} \{(\lambda - \lambda_2)\zeta_m(\lambda)\} \\ &= \lim_{\lambda \rightarrow \lambda_2} \frac{-(\lambda - \lambda_2)}{(\lambda + m - \lambda_2)(\lambda - \lambda_2)} \sum_{k=0}^{m-1} r_{m,k}(\lambda)\zeta_k(\lambda) \\ &= -\frac{1}{m} \sum_{k=0}^{m-1} ((\lambda_2 + k)p_{m-k} + q_{m-k}) a_k. \end{aligned}$$

This completes the proof.  $\square$

Thus, in the case that  $p(z) = 0$  and  $m = 1, 2$ , or  $3$ , the solutions of  $z^2 u''(z) + q(z)u(z) = 0$  have no log-term if and only if

$$(A.14) \quad q_1 = 0 \quad (m = 1) ,$$

$$(A.15) \quad q_2 + (q_1)^2 = 0 \quad (m = 2) ,$$

$$(A.16) \quad q_3 + q_1 q_2 + \frac{1}{4}(q_1)^3 = 0 \quad (m = 3) ,$$

where  $q(z) = \sum_{j=0}^{\infty} q_j z^j$ , as in (A.3).

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